Combinatorial Decision Making and Optimization (Module 2)

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Alma Mater Studiorum · University of Bologna

Contents

1 Satisfiability modulo theory

Satisfiability modulo theory (SMT) Satisfiability of a formula with respect to some background formal theory/theories.

Satisfiability modulo theory (SMT)

SMT extends SAT and exploits domain-specific reasoning (possibly with infinite domains).

1.1 First-order logic for SMT

1.1.1 Syntax

Remark. Only quantifier-free formulas (q.f.f.) are considered in SMT.

Functions The set of all the functions is denoted as $\Sigma^F = \bigcup_{k \geq 0} \Sigma^F_k$ where Σ^F_k denotes the set of k-ary functions.

Functions

Constants Σ^F_0

Predicates The set of all the predicates is denoted as $\Sigma^P = \bigcup_{k \geq 0} \Sigma^P_k$ where Σ^P_k denotes the set of k-ary predicates.

Predicates

Propositional symbols Σ^P_0

Signature The set of the non-logical symbols of FOL is denoted as:

Signature

$$\Sigma = \Sigma^F \cup \Sigma^P$$

Terms The set of terms over Σ is denoted as $\mathbb{T}^\Sigma$:

Terms

$$\begin{aligned}\mathbb{T}^\Sigma = \; & \Sigma^F_0 \cup \\ & \{ f(t_1, \dots, t_k) \mid f \in \Sigma^F_k \wedge t_1, \dots, t_k \in \mathbb{T}^\Sigma \} \cup \\ & \{ \texttt{ite}(\varphi, t_1, t_2) \mid \varphi \in \mathbb{F}^\Sigma \wedge t_1, t_2 \in \mathbb{T}^\Sigma \}\end{aligned}$$

Remark. `ite` is an auxiliary function to capture the if-then-else construct.

Formulas The set of formulas over Σ is denoted as $\mathbb{F}^\Sigma$:

Formulas

$$\begin{aligned}\mathbb{F}^\Sigma = \; & \{\bot, \top\} \cup \Sigma^P_0 \cup \\ & \{ t_1 = t_2 \mid t_1, t_2 \in \mathbb{T}^\Sigma \} \cup \\ & \{ p(t_1, \dots, t_k) \mid p \in \Sigma^P_k \wedge t_1, \dots, t_k \in \mathbb{T}^\Sigma \} \cup \\ & \{ \neg \varphi \mid \varphi \in \mathbb{F}^\Sigma \} \cup \\ & \{ (\varphi_1 \Rightarrow \varphi_2), (\varphi_1 \iff \varphi_2), (\varphi_1 \wedge \varphi_2), (\varphi_1 \vee \varphi_2) \mid \varphi_1, \varphi_2 \in \mathbb{F}^\Sigma \}\end{aligned}$$

1.1.2 Semantics

Σ-model Pair $\mathcal{M} = \langle M, (\cdot)^\mathcal{M} \rangle$ defined on a given signature Σ where: Σ-model

- M is the universe of $\mathcal{M}$.
- $(\cdot)^\mathcal{M}$ is a mapping such that:
  - $\forall f \in \Sigma_k^F : f^\mathcal{M} \in \{\varphi \mid \varphi : M^k \to M\}$.
  - $\forall p \in \Sigma_k^P : p^\mathcal{M} \in \{\varphi \mid \varphi : M^k \to \{\texttt{true}, \texttt{false}\}\}$.

Interpretation Extension of the mapping function $(\cdot)^\mathcal{M}$ to terms and formulas: Interpretation

- $\top^\mathcal{M} = \texttt{true}$ and $\bot^\mathcal{M} = \texttt{false}$.
- $(f(t_1, \dots, t_k))^\mathcal{M} = f^\mathcal{M}(t_1^\mathcal{M}, \dots, t_k^\mathcal{M})$ and $(p(t_1, \dots, t_k))^\mathcal{M} = p^\mathcal{M}(t_1^\mathcal{M}, \dots, t_k^\mathcal{M})$.
- $\texttt{ite}(\varphi, t_1, t_2)^\mathcal{M} = \begin{cases} t_1^\mathcal{M} & \text{if } \varphi^\mathcal{M} = \texttt{true} \\ t_2^\mathcal{M} & \text{if } \varphi^\mathcal{M} = \texttt{false} \end{cases}$.

1.1.3 Σ-theory

Satisfiability A model $\mathcal{M}$ satisfies a formula $\varphi \in \mathbb{F}^\Sigma$ if $\varphi^\mathcal{M} = \texttt{true}$. Satisfiability

Σ-theory Possibly infinite set $\mathcal{T}$ of Σ-models. Σ-theory

$\mathcal{T}$-satisfiability A formula $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-satisfiable if there exists a model $\mathcal{M} \in \mathcal{T}$ that satisfies it. $\mathcal{T}$-satisfiability

$\mathcal{T}$-consistency A set of formulas $\{\varphi_1, \dots, \varphi_k\} \subseteq \mathbb{F}^\Sigma$ is $\mathcal{T}$-consistent iff $\varphi_1 \land \dots \land \varphi_k$ is $\mathcal{T}$-satisfiable. $\mathcal{T}$-consistency

$\mathcal{T}$-entailment A set of formulas $\Gamma \subseteq \mathbb{F}^\Sigma$ $\mathcal{T}$-entails a formula $\varphi \in \mathbb{F}^\Sigma$ ($\Gamma \models_\mathcal{T} \varphi$) iff in every model $\mathcal{M} \in \mathcal{T}$ that satisfies Γ, φ is also satisfied. $\mathcal{T}$-entailment

Remark. Γ is $\mathcal{T}$-consistent iff $\Gamma \not\models_\mathcal{T} \bot$.

$\mathcal{T}$-validity A formula $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-valid iff $\varnothing \models_\mathcal{T} \varphi$. $\mathcal{T}$-validity

Remark. φ is $\mathcal{T}$-consistent iff $\neg\varphi$ is not $\mathcal{T}$-valid.

Theory lemma $\mathcal{T}$-valid clause $c = l_1 \lor \dots \lor l_k$. Theory lemma

Σ-expansion Given a Σ-model $\mathcal{M} = \langle M, (\cdot)^\mathcal{M} \rangle$ and $\Sigma' \supseteq \Sigma$, an expansion $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ over Σ' is any Σ'-model such that: Σ-expansion

- $M' = M$.
- $\forall s \in \Sigma : s^{\mathcal{M}'} = s^\mathcal{M}$

Remark. Given a Σ-theory $\mathcal{T}$, we implicitly consider it to be the theory $\mathcal{T}'$ defined as:

$$\mathcal{T}' = \{\mathcal{M}' \mid \mathcal{M}' \text{ is an expansion of a } \Sigma\text{-model } \mathcal{M} \text{ in } \mathcal{T}\}$$

Ground $\mathcal{T}$-satisfiability Given a Σ-theory $\mathcal{T}$, determine if a ground formula is $\mathcal{T}$-satisfiable over a Σ-expansion $\mathcal{T}'$. Ground $\mathcal{T}$-satisfiability

Axiomatically defined theory Given a minimal set of formulas (axioms) $\Lambda \subseteq \mathbb{F}^\Sigma$, its corresponding theory is the set of all the models that respect Λ. Axiomatically defined theory

Example. Let Σ be defined as:

$$\Sigma_0^F = \{a, b, c, d\} \qquad \Sigma_1^F = \{f, g\} \qquad \Sigma_2^P = \{p\}$$

A Σ-model $\mathcal{M} = \langle [0, 2\pi[, (\cdot)^{\mathcal{M}} \rangle$ can be defined as follows:

$$\begin{gathered} a^{\mathcal{M}} = 0 \qquad b^{\mathcal{M}} = \frac{\pi}{2} \qquad c^{\mathcal{M}} = \pi \qquad d^{\mathcal{M}} = \frac{3\pi}{2} \\ f^{\mathcal{M}} = \sin \qquad g^{\mathcal{M}} = \cos \qquad p^{\mathcal{M}}(x, y) \iff x > y \end{gathered}$$

To determine if $p(g(x), f(d))$ is $\mathcal{M}$-satisfiable, we have to expand $\mathcal{M}$ as there are free variables (x). Let $\Sigma' = \Sigma \cup \{x\}$. The expansion $\mathcal{M}'$ such that $x^{\mathcal{M}'} = \frac{\pi}{2}$ makes the formula satisfiable.

1.1.4 Theories of interest

Equality with Uninterpreted Functions theory (EUF) Theory $\mathcal{T}_{\text{EUF}}$ containing all the possible Σ-models.

Equality with Uninterpreted Functions theory (EUF)

Remark. Also called empty theory as its axiom set is $\varnothing$ (i.e. allows any model).

Remark. Useful to deal with black-box functions (i.e. prove satisfiability without a specific theory).

Example. The following formula can be proved to be unsatisfiable by only using syntactic manipulations of basic FOL concepts:

$$\begin{gathered} \big(a * (f(b) + f(c)) = d\big) \land \big(b * (f(a) + f(c)) \neq d\big) \land \underline{\big(a = b\big)} \\ \big(\underline{a * (f(a) + f(c))} = d\big) \land \big(\underline{a * (f(a) + f(c))} \neq d\big) \\ \big(g(a, c) = d\big) \land \big(g(a, c) \neq d\big) \end{gathered}$$

Arithmetic theories Theories with $\Sigma = (0, 1, +, -, \leq)$.

Arithmetic theories

Presburger arithmetic Theory $\mathcal{T}_{\mathbb{Z}}$ that interprets Σ-symbols over integers.

- Ground $\mathcal{T}_{\mathbb{Z}}$-satisfiability is **NP**-complete.
- Extended with multiplication, $\mathcal{T}_{\mathbb{Z}}$-satisfiability becomes undecidable.

Real arithmetic Theory $\mathcal{T}_{\mathbb{R}}$ that interprets Σ-symbols over reals.

- Ground $\mathcal{T}_{\mathbb{R}}$-satisfiability is in **P**.
- Extended with multiplication, $\mathcal{T}_{\mathbb{R}}$-satisfiability becomes doubly-exponential.

Remark. In floating points, commutativity still holds, but associativity and distributivity are not guaranteed.

Array theory Let $\Sigma_{\mathcal{A}}$ be the signature containing two functions:

Array theory

$\texttt{read}(a, i)$ Reads the value of a at index i.

$\texttt{write}(a, i, v)$ Returns an array a' where the value v is at the index i of a.

The theory $\mathcal{T}_{\mathcal{A}}$ is the set of all models respecting the following axioms:

- $\forall a \, \forall i \, \forall v : \texttt{read}(\texttt{write}(a, i, v), i) = v$.
- $\forall a \, \forall i \, \forall j \, \forall v : (i \neq j) \Rightarrow \Big(\texttt{read}\big(\texttt{write}(a, i, v), j\big) = \texttt{read}(a, j)\Big)$.
- $\forall a \, \forall a' : \big(\forall i : \texttt{read}(a, i) = \texttt{read}(a', i)\big) \Rightarrow (a = a')$.

Remark. The full $\mathcal{T}_\mathcal{A}$ theory is undecidable but there are decidable fragments.

Bit-vectors theory Theory $\mathcal{T}_{\mathcal{BV}}$ with vectors of bits of fixed length as constants and operations such as: Bit-vectors theory

- String-like operations (e.g. slicing, concatenation, ...).
- Logical operations (e.g. bit-wise operators).
- Arithmetic operations (e.g. $+$, $-$, ...).

String theory Theory to handle strings of unbounded length. String theory

Theory of word equations Given an alphabet $\mathcal{S}$, a word equation has form $L = R$ where L and R are concatenations of string constants over $\mathcal{S}^*$.

Remark. The general theory of word equations is undecidable.

Remark. The quantifier-free theory of word equations is decidable.

Remark. In practice, many theories are often combined.

1.2 Encoding to SAT

1.2.1 Eager approaches

All the information on the formal theory is used from the beginning to encode an SMT formula φ into an equisatisfiable SAT formula φ' (i.e. SMT is compiled into SAT).

Equisatisfiability Given a Σ-theory $\mathcal{T}$, two formulas φ and φ' are equisatisfiable iff: Equisatisfiability

$$\varphi \text{ is } \mathcal{T}\text{-satisfiable} \iff \varphi' \text{ is } \mathcal{T}\text{-satisfiable}$$

Eager approaches have the following advantages:

- Does not require an SMT solver.
- Once encoded, whichever SAT solver can be used.

Eager approaches have the following disadvantages:

- An ad-hoc encoding is needed for all the theories.
- The resulting SAT formula might be huge.

Algorithm Given an EUF formula φ, to determine if it is $\mathcal{T}_{\text{EUF}}$-satisfiable, the following steps are taken:

1. Replace functions and predicates with constant equalities. Given the terms $f(t_1), \dots, f(t_k)$, possible approaches are:

   Ackermann approach Ackermann approach

   - Each $f(t_i)$ is encoded into a new constant A_i.
   - Add the constraints $(t_i = t_j) \Rightarrow (A_i = A_j)$ for each $i < j$.

   Bryant approach Bryant approach

   - $f(t_1)$ is encoded as A_1.
   - $f(t_2)$ is encoded as $\texttt{ite}(t_2 = t_1, A_1, A_2)$.

- $f(t_3)$ is encoded as $\texttt{ite}\big(t_3 = t_1, A_1, \texttt{ite}(t_3 = t_2, A_2, A_3)\big)$.
- $f(t_i)$ is encoded as:

$$\texttt{ite}\Big(t_i = t_1, A_1, \texttt{ite}\Big(t_i = t_2, A_2, \texttt{ite}\big(\dots, \texttt{ite}(t_i = t_{i-1}, A_{i-1}, A_i)\big)\Big)\Big)$$

2. Remove equalities to reduce φ into propositional logic. Possible encodings are:

   Small-domain encoding If φ has n distinct variables $\{c_1, \dots, c_n\}$, a possible model $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ that satisfies it must have $|M| \leq n$.

   Therefore, each $c_i^{\mathcal{M}}$ can be associated to a value in $\{1, \dots, n\}$. In SAT, this mapping from $c_i^{\mathcal{M}}$ to $\{1, \dots, n\}$ can be encoded using $O(\log n)$ bits. Finally, an equality $c_i = c_j$ (or $c_i \neq c_j$) can be encoded by adding bitwise constraints.

   Direct encoding Encode each equality $a = b$ with a propositional symbol $P_{a,b}$ and add transitivity constraints of form $(P_{a,b} \land P_{b,c}) \Rightarrow P_{a,c}$.

1.2.2 Lazy approaches

Integrate SAT solvers with theory-specific decision procedures.
These approaches are more flexible and modular and avoid an explosion of SAT clauses.
On the other hand, the search becomes SAT-driven and not theory-driven.

Remark. Most SMT solvers follow a lazy approach.

Algorithm Let $\mathcal{T}$ be a theory. Given a conjunction of $\mathcal{T}$-literals $\varphi = \varphi_1 \land \dots \land \varphi_n$, to determine its $\mathcal{T}$-satisfiability, a generic lazy solver does the following:

1. Each SMT literal φ_i is encoded (abstracted) into a SAT literal l_i to form the abstraction $\Phi = \{l_1, \dots, l_n\}$ of φ.
2. The $\mathcal{T}$-solver sends Φ to the SAT-solver.
   - If the SAT-solver determines that Φ is unsatisfiable, then φ is $\mathcal{T}$-unsatisfiable.
   - Otherwise, the SAT-solver returns a model $\mathcal{M} = \{a_1, \dots, a_n\}$ (an assignment of the literals, possibly partial).
3. The $\mathcal{T}$-solver determines if $\mathcal{M}$ is $\mathcal{T}$-consistent.
   - If it is, then φ is $\mathcal{T}$-satisfiable.
   - Otherwise, update $\Phi = \Phi \cup \neg\mathcal{M}$ and go to Point 2.

Example. Consider the EUF formula φ:

$$\big(g(a) = c\big) \land \big((f(g(a)) \neq f(c)) \lor (g(a) = d)\big) \land \big(c \neq d\big)$$

- φ abstracted into SAT is:

$$\underbrace{\big(g(a) = c\big)}_{l_1} \land \big(\neg \underbrace{(f(g(a)) = f(c))}_{l_2} \lor \underbrace{(g(a) = d)}_{l_3}\big) \land \neg \underbrace{(c = d)}_{l_4}$$

$$l_1 \land (\neg l_2 \lor l_3) \land \neg l_4$$

  Therefore, $\Phi = \{l_1, (\neg l_2 \lor l_3), \neg l_4\}$
- The $\mathcal{T}$-solver sends Φ to the SAT-solver. Let's say that it return $\mathcal{M} = \{l_1, \neg l_2, \neg l_4\}$.

- The $\mathcal{T}$-solver checks if $\mathcal{M}$ is consistent. Let's say it is not. Let $\Phi' = \Phi \cup \neg\mathcal{M} = \{l_1, (\neg l_2 \vee l_3), \neg l_4, (\neg l_1 \vee l_2 \vee l_4)\}$.
- The $\mathcal{T}$-solver sends Φ' to the SAT-solver. Let's say that it return $\mathcal{M}' = \{l_1, l_2, l_3, \neg l_4\}$.
- The $\mathcal{T}$-solver checks if $\mathcal{M}'$ is consistent. Let's say it is not. Let $\Phi'' = \Phi' \cup \neg\mathcal{M}' = \{l_1, (\neg l_2 \vee l_3), \neg l_4, (\neg l_1 \vee l_2 \vee l_4), (\neg l_1 \vee \neg l_2 \vee \neg l_3 \vee l_4)\}$.
- The $\mathcal{T}$-solver sends Φ'' to the SAT-solver and it detects the unsatisfiability. Therefore, φ is $\mathcal{T}$-unsatisfiable.

Optimizations

- Check $\mathcal{T}$-consistency on partial assignments.
- Given a $\mathcal{T}$-inconsistent assignment μ, find a smaller $\mathcal{T}$-inconsistent assignment $\eta \subseteq \mu$ and add $\neg\eta$ to Φ instead of $\neg\mu$.
- When reaching $\mathcal{T}$-inconsistency, backjump to a $\mathcal{T}$-consistent point in the computation.

1.3 CDCL($\mathcal{T}$)

Lazy solver based on CDCL for SAT extended with a $\mathcal{T}$-solver. The $\mathcal{T}$-solver does the following: CDCL($\mathcal{T}$)

- Checks the $\mathcal{T}$-consistency of a conjunction of literals.
- Performs deduction of unassigned literals.
- Explains $\mathcal{T}$-inconsistent assigments.
- Allows to backtrack.

State transition Transition system to describe the reasoning of SAT or SMT solvers. A transition has form: State transition

$$(\mu \| \varphi) \to (\mu' \| \varphi')$$

where:

- φ and φ' are $\mathcal{T}$-formulas.
- μ and μ' are (partial) boolean assignments to atoms of φ and φ', respectively.
- $(\mu \| \varphi)$ and $(\mu' \| \varphi')$ are states.

Transition rule Determine the possible transitions.

Derivation Sequence of transitions.

Initial state $(\varnothing \| \varphi)$.

$\mathcal{T}$-consistency Given a $\mathcal{T}$-formula φ and a full assignment μ of φ, φ is $\mathcal{T}$-consistent ($\mu \models_{\mathcal{T}} \varphi$) if there is a derivation from $(\varnothing \| \varphi)$ to $(\mu \| \varphi)$.

$\mathcal{T}$-propagation Deduce the assignment of an unassigned literal l using some knowledge of the theory. $\mathcal{T}$-propagation

$\mathcal{T}$-consequence If: $\mathcal{T}$-consequence

- $\mu \models_{\mathcal{T}} l$,

- l or $\neg l$ occur in φ,
- l and $\neg l$ do not occur in μ,

then:

$$(\mu \| \varphi) \rightarrow (\mu \cup \{l\} \| \varphi)$$

Example. Given the formula φ:

$$\big(g(a) = c\big) \wedge \Big(\big(f(g(a)) \neq f(c)\big) \vee \big(g(a) = d\big)\Big) \wedge \big(c \neq d\big)$$

A possible derivation for some theory $\mathcal{T}$ (i.e. $\mathcal{T}$-propagation are decided arbitrarily) is:

1. $\varnothing \| \varphi$ (initial state).
2. $\varnothing \| \varphi \rightarrow \{l_1\} \| \varphi$ (Unit propagation).
3. $\{l_1\} \| \varphi \rightarrow \{l_1, l_2\} \| \varphi$ ($\mathcal{T}$-propagation).
4. $\{l_1, l_2\} \| \varphi \rightarrow \{l_1, l_2, l_3\} \| \varphi$ (Unit propagation).
5. $\{l_1, l_2, l_3\} \| \varphi \rightarrow \{l_1, l_2, l_3, l_4\} \| \varphi$ ($\mathcal{T}$-propagation).
6. $\{l_1, l_2, l_3, l_4\} \| \varphi \rightarrow$ `fail` (Failure).

As we are at decision level 0 (as no decision literal has been fixed), we can conclude that φ is unsatisfiable.

Remark. Unit and theory propagation are alternated (see algorithm description).

Algorithm Given a $\mathcal{T}$-formula φ and a (partial) $\mathcal{T}$-assignment μ (i.e. initial decisions), CDCL($\mathcal{T}$) does the following:

Algorithm 1 CDCL($\mathcal{T}$)

```
def cdclT(φ, μ):
    if preprocess(φ, μ) == CONFLICT: return UNSAT
    φ^p, μ^p = SMT_to_SAT(φ), SMT_to_SAT(μ)
    level = 0
    l = None

    while True:
        status = propagate(φ^p, μ^p, l)
        if status == SAT:
            return SAT_to_SMT(μ^p)
        elif status == UNSAT:
            η^p, jump_level = analyzeConflict(φ^p, μ^p)
            if jump_level < 0: return UNSAT
            backjump(jump_level, φ^p ∧ ¬η^p, μ^p)
        elif status == UNKNOWN:
            l = decideNextLiteral(φ^p, μ^p)
            level += 1
```

Where:

`preprocess` Preprocesses φ with μ through operations such as simplifications, $\mathcal{T}$-specific rewritings, ...

`SMT_TO_SAT` Provides the boolean abstraction of an SMT formula.

`SAT_TO_SMT` Reverses the boolean abstraction of an SMT formula.

`propagate` Iteratively apply:

- Unit propagation,
- $\mathcal{T}$-consistency check,
- $\mathcal{T}$-propagation.

Returns `SAT`, `UNSAT` or `UNKNOWN` (when no deductions are possible and there are still free variables).

`analyzeConflict` Performs conflict analysis:

- If the conflict is detected by SAT boolean propagation ($\mu^p \wedge \varphi^p \models_p \bot$), a boolean conflict set η^p is outputted (as in standard CDCL).
- If the conflict is detected by $\mathcal{T}$-propatation ($\mu \wedge \phi \models_{\mathcal{T}} \bot$), a theory conflict η is produced and its boolean abstraction η^p is outputted.

Moreover, the earliest decision level at which a variable of η^p is unassigned is returned.

As in standard CDCL, $\neg\eta^p$ is added to φ^p and the algorithm backjumps to a previous decision level (if possible).

`decideNextLiteral` Decides the assignment of an unassigned variable (as in standard CDCL). Theory information might be exploited.

Implication graph As in the standard CDCL algorithm, an implication graph is used to explain conflicts. Implication graph

Nodes Decisions, derived literals or conflicts.

Edges If v allows to unit/theory propagate w, then there is an edge $v \rightarrow w$.

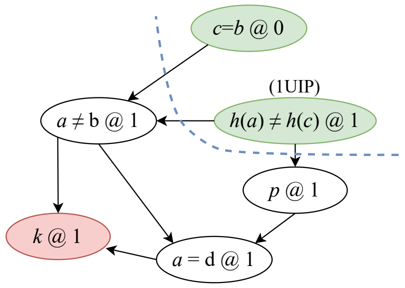
Example. Given the $\mathcal{T}$-formula φ:

$$\big(h(a) = h(c) \vee p\big) \wedge \big(a = b \vee \neg p \vee a = d\big) \wedge \big(a \neq d \vee a = b\big)$$

and an initial decision $(c = b) \in \mu$, CDCL($\mathcal{T}$) does the following:

1. As no propagation is possible, the decision $h(a) \neq h(c)$ is added to μ.
2. Unit propagate p due to the clause $\big(h(a) = h(c) \vee p\big)$ and the decision at the previous step.
3. $\mathcal{T}$-propagate $(a \neq b)$ due to the current assigments: $\{c = b, h(a) \neq h(c)\} \models_{\mathcal{T}} a \neq b$.
4. Unit propagate $(a = d)$ due to the clause $\big(a = b \vee \neg p \vee a = d\big)$ and the current knowledge base (p and $a \neq b$).
5. There is a conflict between $(a \neq d)$ and $(a = d)$.

By building the conflict graph, one can identify the 1UIP as the decision $h(a) \neq h(c)$.



A cut in front of the 1UIP that separates decision nodes and the conflict node (as in standard CDCL) is made to obtain the conflict set $\eta = \{h(a) \neq h(c), c = b\}$. $\big((h(a) = h(c)) \vee (c \neq b)\big)$ is added as a clause and the algorithm backjumps at the decision level 0.

1.4 Theory solvers

Decide satisfiability of theory-specific formulas.

1.4.1 EUF theory

Congruence closure Given a conjunction of EUF literals Φ, its satisfiability can be decided in polynomial time as follows:

Congruence closure

1. Add a new variable c and replace each $p(t_1, \dots, t_k)$ with $f_p(t_1, \dots, t_k) = c$.
2. Partition input literals into the sets of equalities E and disequalities D.
3. Define E^* as the congruence closure of E. It is the smallest equivalence relation $\equiv_E$ over terms such that:
   - $(t_1 = t_2) \in E \Rightarrow (t_1 \equiv_E t_2)$.
   - For each $f(s_1, \dots, s_k)$ and $f(t_1, \dots, t_k)$ occurring in E, if $s_i \equiv_E t_i$ for each $i \in \{1, \dots, k\}$, then $f(s_1, \dots, s_k) \equiv_E f(t_1, \dots, t_k)$.
4. Φ is satisfiable iff $\forall (t_1 \neq t_2) \in D \Rightarrow (t_1 \not\equiv_E t_2)$.

Remark. In practice, congruence closure is usually implemented using a DAG to represent terms and union-find for the E^* class.

1.4.2 Arithmetic theories

Linear real arithmetic LRA theory has signature $\Sigma_{\text{LRA}} = (\mathbb{Q}, +, -, *, \leq)$ where the multiplication $*$ is only linear.

Fourier-Motzkin elimination Given an LRA formula, its satisfiability can be decided as follows:

Fourier-Motzkin elimination

1. Replace:
   - $(t_1 \neq t_2)$ with $(t_1 < t_2) \vee (t_2 < t_1)$.
   - $(t_1 \leq t_2)$ with $(t_1 < t_2) \vee (t_1 = t_2)$.
2. Eliminate equalities and apply the Fourier-Motzkin elimination[1] method on all variables to determine satisfiability.

Remark. Not practical on a large number of constraints. The simplex algorithm is more suited.

Linear integer arithmetic LIA theory has signature $\Sigma_{\text{LRA}} = (\mathbb{Z}, +, -, *, \leq)$ where the multiplication $*$ is only linear.

Fourier-Motzkin can be applied to check satisfiability. Simplex and branch & bound is usually better.

[1] https://en.wikipedia.org/wiki/Fourier%E2%80%93Motzkin_elimination

1.4.3 Difference logic theory

Difference logic (DL) atomic formulas have form $(x - y \leq k)$ where x, y are variables and k is a constant.

Remark. Constraints with form $(x - y \bowtie k)$ where $\bowtie \in \{=, \neq, <, \geq, >\}$ can be rewritten using $\leq$.

Remark. Unary constraints $x \leq k$ can be rewritten as $x - z_0 \leq k$ with the assignment $z_0 = 0$.

Theorem 1.4.1. By allowing $\neq$ and with domain in $\mathbb{Z}$, deciding satisfiability becomes NP-hard.

Proof. Any graph k-coloring instance can be poly-reduced to a difference logic formula. □

Graph consistency Given DL literals Φ, it is possible to build a weighted graph $\mathcal{G}_\Phi$ where: Graph consistency

Nodes Variables occurring in Φ.

Edges $x \xrightarrow{k} y$ for each $(x - y \leq k) \in \Phi$.

Theorem 1.4.2. Φ is inconsistent $\iff$ $\mathcal{G}_\Phi$ has a negative cycle (i.e. cycle whose cost is negative).

Remark. A negative cycle acts as an inconsistency explanation (not necessarily minimal).

Remark. From the consistency graph, if there is a path from x to y with cost k, then $(x - y \leq k)$ can be deduced.

1.5 Combining theories

Given $\mathcal{T}_i$-solvers for theories $\mathcal{T}_1, \dots, \mathcal{T}_n$, a general approach to combine them into a $\bigcup_i^n \mathcal{T}_i$-solver is the following:

1. Purify the formula so that each literal belongs to a single theory. New constants can be introduced.

   Interface equalities Equalities involving shared constants across solvers should be the same for all solvers.

2. Iteratively run the following:
   a) Each $\mathcal{T}_i$-solver checks the satisfiability of $\mathcal{T}_i$-formulas. If one detects unsatisfiability, the whole formula is unsatisfiable.
   b) When a $\mathcal{T}_i$-solver deduces a new literal, it sends it to the other solvers.

Example. Consider the formula:

$$\Big(f\big(f(x) - f(y)\big) = a\Big) \land \Big(f(a) = a + 2\Big) \land \Big(x = y\Big)$$

where the theories of EUF and linear arithmetic (LA) are involved.
To determine satisfiability, the following steps are taken:

1. The formula is purified to obtain the literals:

LA	EUF
	$f(e_1) = a$
$e_1 = e_2 - e_3$	$e_2 = f(x)$
$e_4 = 0$	$e_3 = f(y)$
$e_5 = a + 2$	$f(e_4) = e_5$
	$x = y$

where $e_1, \dots, e_5$ are new constants.

2. Both EUF-solver and LA-solver determine `SAT`. Moreover, the EUF-solver deduces that $\{x = y, f(x) = e_2, f(y) = e_3\} \models_{EUF} (e_2 = e_3)$ and sends it to the LA-solver.

LA	EUF
$e_1 = e_2 - e_3$	$f(e_1) = a$
$e_4 = 0$	$e_2 = f(x)$
$e_5 = a + 2$	$e_3 = f(y)$
$\underline{e_2 = e_3}$	$f(e_4) = e_5$
	$x = y$

3. Both EUF-solver and LA-solver determine `SAT`. Moreover, the LA-solver deduces that $\{e_2 - e_3 = e_1, e_4 = 0, e_2 = e_3\} \models_{LA} (e_1 = e_4)$ and sends it to the EUF-solver.

LA	EUF
	$f(e_1) = a$
$e_1 = e_2 - e_3$	$e_2 = f(x)$
$e_4 = 0$	$e_3 = f(y)$
$e_5 = a + 2$	$f(e_4) = e_5$
$e_2 = e_3$	$x = y$
	$\underline{e_1 = e_4}$

$$\vdots$$

4. The EUF-solver determines `SAT` but the LA-solver determines `UNSAT`. Therefore, the formula is unsatisfiable.

1.5.1 Deterministic Nelson-Oppen

Deterministic Nelson-Oppen

Let $\mathcal{T}_1$ be a Σ_1-theory and $\mathcal{T}_2$ be a Σ_2-theory. $(\mathcal{T}_1 \cup \mathcal{T}_2)$-satisfiability can be checked with the deterministic Nelson-Oppen if $\mathcal{T}_1$ and $\mathcal{T}_2$ are:

Signature-disjoint $\Sigma_1 \cap \Sigma_2 = \varnothing$.

Stably-infinite $\mathcal{T}_i$ is stably-infinite iff every $\mathcal{T}_i$-satisfiable formula φ has a corresponding $\mathcal{T}_i$-model with a universe of infinite cardinality that satisfies it.

Convex For each set of $\mathcal{T}_i$-literals S, it holds that:

$$\big(S \models_{\mathcal{T}_i} (a_1 = b_1) \vee \dots \vee (a_n = b_n)\big) \Rightarrow \big(S \models_{\mathcal{T}_i} (a_k = b_k)\big) \text{ for some } k \in \{1, \dots, n\}$$

Example. $\mathcal{T}_\mathbb{Z}$ is not convex. Consider the following formula φ:

$$(1 \leq z) \wedge (z \leq 2) \wedge (u = 1) \wedge (v = 2)$$

We have that:

$$\varphi \models_{\mathcal{T}_\mathbb{Z}} (z = u) \vee (z = v)$$

But it is not true that:

$$\varphi \not\models_{\mathcal{T}_{\mathbb{Z}}} z = u \qquad \varphi \not\models_{\mathcal{T}_{\mathbb{Z}}} z = v$$

Algorithm Given a $(\mathcal{T}_1 \cup \mathcal{T}_2)$-formula S, the deterministic Nelson-Oppen algorithm works as follows:

1. Purify S into S_1 and S_2. Let E be the set of interface equalities between S_1 and S_2 (i.e. it contains all the equalities that involve shared constants).
2. If $S_1 \models_{\mathcal{T}_1} \perp$ or $S_2 \models_{\mathcal{T}_2} \perp$, then S is unsatisfiable.
3. If $S_1 \models_{\mathcal{T}_1} (x = y)$ with $(x = y) \in (E \setminus S_2)$, then $S_2 \leftarrow S_2 \cup \{x = y\}$. Go to Point 2.
4. If $S_2 \models_{\mathcal{T}_2} (x = y)$ with $(x = y) \in (E \setminus S_1)$, then $S_1 \leftarrow S_1 \cup \{x = y\}$. Go to Point 2.
5. S is satisfiable.

1.5.2 Non-deterministic Nelson-Oppen

Extension of the deterministic Nelson-Oppen algorithm to non-convex theories.
Works by doing case splitting on pairs of shared variables and has exponential time complexity.

Non-deterministic Nelson-Oppen

1.6 SMT extensions

1.6.1 Layered solvers

Stratify the problem into layers of increasing complexity. The satisfiability of each layer is determined by a different solver of increasing expressivity and complexity.

Layered solvers

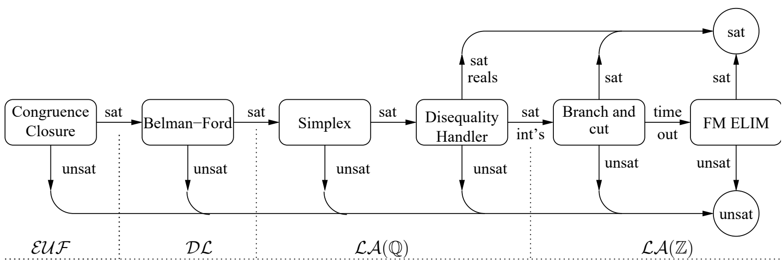


Figure 1.1: Example of layered solvers

1.6.2 Case splitting

Case reasoning on free variables.

Case splitting

Example. Given the formula:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big)$$

A solver can explore the case when $i = j$ and $i \neq j$.

$\mathcal{T}$-solver case reasoning The $\mathcal{T}$ solver internally detects inconsistencies through case reasoning.

SAT solver case reasoning The $\mathcal{T}$-solver encodes the case reasoning and sends it to the SAT solver.

Example. Given the formula:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big)$$

The $\mathcal{T}$-solver sends to the SAT solver the following:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big) \wedge (i = j) \Rightarrow y = x$$
$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big) \wedge (i \neq j) \Rightarrow y = \texttt{read}(A, j)$$

1.6.3 Optimization modulo theory

Extension of SMT so that it finds a model that simultaneously satisfies the input formula φ and optimizes an objective function f_{obj}. Optimization modulo theory
φ belongs to a theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ where $\mathcal{T}_{\preceq}$ contains a predicate $\preceq$ (e.g. $\leq$) representing a total order.

Offline OTM($\mathcal{LRA}$) Approach that does not require to modify the SMT solver.

- **Linear search** Repeatedly solve the problem and, at each iteration, add the constraint $\texttt{cost} < c_i$ where c_i is the cost found at the i-th iteration.
- **Binary search** Given the cost domain $[l_i, u_i]$, repeatedly pick a pivot $p_i \in [l_i, u_i]$ and add the constraint $\texttt{cost} < p_i$. If a model is found, recurse in the domain $[l_i, p_i]$, otherwise recurse in $[p_i, u_i]$.

Inline OTM($\mathcal{LRA}$) SMT solver that integrates an optimization procedure.