Languages and Algorithms for Artificial Intelligence (Module 2)

Last update: 22 December 2023

Academic Year 2023 – 2024
Alma Mater Studiorum · University of Bologna

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1 Propositional logic

1.1 Syntax

Syntax Rules and symbols to define well-formed sentences. Syntax

The symbols of propositional logic are:

Proposition symbols p_0, p_1, ...

Connectives $\wedge \vee \rightarrow \leftrightarrow \neg \perp (\,)$

Well-formed formula The definition of a well-formed formula is recursive: Well-formed formula

- An atomic proposition is a well-formed formula.
- If S is well-formed, $\neg S$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \wedge S_2$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \vee S_2$ is well-formed.

Note that the implication $S_1 \rightarrow S_2$ can be written as $\neg S_1 \vee S_2$.

The BNF definition of a formula is:

$$F := \texttt{atomic\_proposition} \,|\, F \wedge F \,|\, F \vee F \,|\, F \rightarrow F \,|\, F \leftrightarrow F \,|\, \neg F \,|\, (F)$$

1.2 Semantics

Semantics Rules to associate a meaning to well-formed sentences. Semantics

Model theory What is true.

Proof theory What is provable.

Interpretation Given a propositional formula F of n atoms $\{A_1, \dots, A_n\}$, an interpretation $\mathcal{I}$ of F is is a pair (D, I) where: Interpretation

- D is the domain. Truth values in the case of propositional logic.
- I is the interpretation mapping that assigns to the atoms $\{A_1, \dots, A_n\}$ an element of D.

Note: given a formula F of n distinct atoms, there are 2^n district interpretations.

Model If F is true under the interpretation $\mathcal{I}$, we say that $\mathcal{I}$ is a model of F ($\mathcal{I} \models F$). Model

Valid formula A formula F is valid (tautology) iff it is true in all the possible interpretations. It is denoted as $\models F$. Valid formula

Invalid formula A formula F is invalid iff it is not valid (:o). Invalid formula

In other words, there is at least an interpretation where F is false.

Inconsistent formula A formula F is inconsistent (unsatisfiable) iff it is false in all the possible interpretations. Inconsistent formula

Consistent formula A formula F is consistent (satisfiable) iff it is not inconsistent. Consistent formula

In other words, there is at least an interpretation where F is true.

Decidability A logic is decidable if there is a terminating method to decide if a formula is valid. Decidability

Propositional logic is decidable.

Truth table Useful to define the semantics of connectives. Truth table

- $\neg S$ is true iff S is false.
- $S_1 \wedge S_2$ is true iff S_1 is true and S_2 is true.
- $S_1 \vee S_2$ is true iff S_1 is true or S_2 is true.
- $S_1 \rightarrow S_2$ is true iff S_1 is false or S_2 is true.
- $S_1 \leftrightarrow S_2$ is true iff $S_1 \rightarrow S_2$ is true and $S_1 \leftarrow S_2$ is true.

Evaluation The connectives of a propositional formula are evaluated in the order: Evaluation order

$$\leftrightarrow, \rightarrow, \vee, \wedge, \neg$$

Formulas in parenthesis have higher priority.

Logical consequence Let $\Gamma = \{F_1, \dots, F_n\}$ be a set of formulas (premises) and G a formula (conclusion). G is a logical consequence of Γ ($\Gamma \models G$) if in all the possible interpretations $\mathcal{I}$, if $F_1 \wedge \dots \wedge F_n$ is true, G is true. Logical consequence

Logical equivalence Two formulas F and G are logically equivalent ($F \equiv G$) iff the truth values of F and G are the same under the same interpretation. In other words, $F \equiv G \iff F \models G \wedge G \models F$. Logical equivalence

Common equivalences are:

Commutativity : $(P \wedge Q) \equiv (Q \wedge P)$ and $(P \vee Q) \equiv (Q \vee P)$

Associativity : $((P \wedge Q) \wedge R) \equiv (P \wedge (Q \wedge R))$ and $((P \vee Q) \vee R) \equiv (P \vee (Q \vee R))$

Double negation elimination : $\neg(\neg P) \equiv P$

Contraposition : $(P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)$

Implication elimination : $(P \rightarrow Q) \equiv (\neg P \vee Q)$

Biconditional elimination : $(P \leftrightarrow Q) \equiv ((P \rightarrow Q) \wedge (Q \rightarrow P))$

De Morgan : $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ and $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Distributivity of $\wedge$ over $\vee$: $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$

Distributivity of $\vee$ over $\wedge$: $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$

1.2.1 Normal forms

Negation normal form (NNF) A formula is in negation normal form iff negations appear only in front of atoms (i.e. not parenthesis). Negation normal form

Conjunctive normal form (CNF) A formula F is in conjunctive normal form iff: Conjunctive normal form

- it is in negation normal form;

- it has the form $F := F_1 \wedge F_2 \cdots \wedge F_n$, where each F_i (clause) is a disjunction of literals.

Example.
$(\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF.
$\neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF (not in NNF).

Disjunctive normal form (DNF) A formula F is in disjunctive normal form iff:

Disjunctive normal form

- it is in negation normal form;
- it has the form $F := F_1 \vee F_2 \cdots \vee F_n$, where each F_i is a conjunction of literals.

1.3 Reasoning

Reasoning method Systems to work with symbols.

Reasoning method

Given a set of formulas Γ, a formula F and a reasoning method E, we denote with $\Gamma \vdash^E F$ the fact that F can be deduced from Γ using the reasoning method E.

Sound A reasoning method E is sound iff:

Soundness

$$(\Gamma \vdash^E F) \rightarrow (\Gamma \models F)$$

Complete A reasoning method E is complete iff:

Completeness

$$(\Gamma \models F) \rightarrow (\Gamma \vdash^E F)$$

Deduction theorem Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G:

Deduction theorem

$$(F_1 \wedge \cdots \wedge F_n) \models G \iff \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$$

Proof.

$\rightarrow$) By hypothesis $(F_1 \wedge \cdots \wedge F_n) \models G$.

So, for each interpretation $\mathcal{I}$ in which $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. Therefore, $\mathcal{I} \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

Moreover, for each interpretation $\mathcal{I}'$ in which $(F_1 \wedge \cdots \wedge F_n)$ is false, $(F_1 \wedge \cdots \wedge F_n) \rightarrow G$ is true. Therefore, $\mathcal{I}' \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

In conclusion, $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

$\leftarrow$) By hypothesis $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$. Therefore, for each interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true.

In conclusion, $(F_1 \wedge \cdots \wedge F_n) \models G$.

$\square$

Refutation theorem Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G:

Refutation theorem

$$(F_1 \wedge \cdots \wedge F_n) \models G \iff F_1 \wedge \cdots \wedge F_n \wedge \neg G \text{ is inconsistent}$$

Note: this theorem is not accepted in intuitionistic logic.

Proof. By definition, $(F_1 \wedge \cdots \wedge F_n) \models G$ iff for every interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. This requires that there are no interpretations where $(F_1 \wedge \cdots \wedge F_n)$ is true and G false. In other words, it requires that $(F_1 \wedge \cdots \wedge F_n \wedge \neg G)$ is inconsistent. $\square$

1.3.1 Natural deduction

Proof theory Set of rules that allows to derive conclusions from premises by exploiting syntactic manipulations.

Proof theory

Natural deduction Set of rules to introduce or eliminate connectives. We consider a subset $\{\wedge, \rightarrow, \bot\}$ of functionally complete connectives.

Natural deduction for propositional logic

Natural deduction can be represented using a tree like structure:

$$\cfrac{\begin{matrix}[\text{hypothesis}]\\ \vdots \\ \text{premise}\end{matrix}}{\text{conclusion}}\ \text{rule name}$$

The conclusion is true when the hypothesis are able to prove the premise. Another tree can be built on top of premises to prove them.

Introduction Usually used to prove the conclusion by splitting it.

Introduction rules

$$\frac{\psi \qquad \varphi}{\varphi \wedge \psi}\ \wedge\text{I} \qquad\qquad \cfrac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi}\ \rightarrow\text{I}$$

Elimination Usually used to exploit hypothesis and derive a conclusion.

Elimination rules

$$\frac{\varphi \wedge \psi}{\varphi}\ \wedge\text{E} \qquad \frac{\varphi \wedge \psi}{\psi}\ \wedge\text{E} \qquad \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \rightarrow\text{E}$$

Ex falso sequitur quodlibet From contradiction, anything follows. This can be used when we have two contradicting hypothesis.

Ex falso sequitur quodlibet

$$\frac{\bot}{\varphi}\ \bot$$

Reductio ad absurdum Assume the opposite and prove a contradiction (not accepted in intuitionistic logic).

Reductio ad absurdum

$$\cfrac{\begin{matrix}[\neg\varphi]\\ \vdots \\ \bot\end{matrix}}{\varphi}\ \text{RAA}$$

2 First order logic

2.1 Syntax

The symbols of propositional logic are: Syntax

Constants Known elements of the domain. Do not represent truth values.

Variables Unknown elements of the domain. Do not represent truth values.

Function symbols Function $f^{(n)}$ applied on n constants to obtain another constant.

Predicate symbols Function $P^{(n)}$ applied on n constants to obtain a truth value.

Connectives $\forall \; \exists \; \wedge \; \vee \; \rightarrow \; \neg \; \leftrightarrow \; \top \; \bot \; (\;)$

Using the basic syntax, the following constructs can be defined:

Term Denotes elements of the domain.

$$t := \texttt{constant} \,|\, \texttt{variable} \,|\, f^{(n)}(t_1, \dots, t_n)$$

Proposition Denotes truth values.

$$P := \top \,|\, \bot \,|\, P \wedge P \,|\, P \vee P \,|\, P \rightarrow P \,|\, P \leftrightarrow P \,|\, \neg P \,|\, \forall x.P \,|\, \exists x.P \,|\, (P) \,|\, P^{(n)}(t_1, \dots, t_n)$$

Well-formed formula The definition of well-formed formula in first order logic extends the one of propositional logic by adding the following conditions: Well-formed formula

- If S is well-formed, $\exists X.S$ is well-formed. Where X is a variable.
- If S is well-formed, $\forall X.S$ is well-formed. Where X is a variable.

Free variables The universal and existential quantifiers bind their variable within the scope of the formula. Let $F_v(F)$ be the set of free variables in a formula F, F_v is defined as follows: Free variables

- $F_v(p(t)) = \bigcup \texttt{vars}(t)$
- $F_v(\top) = F_v(\bot) = \varnothing$
- $F_v(\neg F) = F_v(F)$
- $F_v(F_1 \wedge F_2) = F_v(F_1 \vee F_2) = F_v(F_1 \rightarrow F_2) = F_v(F_1) \cup F_v(F_2)$
- $F_v(\forall X.F) = F_v(\exists X.F) = F_v(F) \setminus \{X\}$

Closed formula/Sentence Proposition without free variables. Sentence

Theory Set of sentences. Theory

Ground term/Formula Proposition without variables. Formula

2.2 Semantics

Interpretation An interpretation in first order logic $\mathcal{I}$ is a pair (D, I): Interpretation

- D is the domain of the terms.
- I is the interpretation function such that:
  - $I(f) : D^n \to D$ for every n-ary function symbol.
  - $I(p) \subseteq D^n$ for every n-ary predicate symbol.

Variable evaluation Given an interpretation $\mathcal{I} = (D, I)$ and a set of variables $\mathcal{V}$, a variable is evaluated through $\eta : \mathcal{V} \to D$. Variable evaluation

Model Given an interpretation $\mathcal{I}$ and a formula F, $\mathcal{I}$ models F ($\mathcal{I} \models F$) when $\mathcal{I}, \eta \models F$ for every variable evaluation η. Model

A sentence S is:

Valid S is satisfied by every interpretation ($\forall \mathcal{I} : \mathcal{I} \models S$).

Satisfiable S is satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \models S$).

Falsifiable S is not satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \not\models S$).

Unsatisfiable S is not satisfied by any interpretation ($\forall \mathcal{I} : \mathcal{I} \not\models S$).

Logical consequence A sentence T_1 is a logical consequence of T_2 ($T_2 \models T_1$) if every model of T_2 is also model of T_1: Logical consequence

$$\mathcal{I} \models T_2 \to \mathcal{I} \models T_1$$

Theorem 2.2.1. It is undecidable to determine if a first order logic formula is a tautology.

Equivalence A sentence T_1 is equivalent to T_2 if $T_1 \models T_2$ and $T_2 \models T_1$. Equivalence

Theorem 2.2.2. The following statements are equivalent:

1. $F_1, \dots, F_n \models G$.
2. $(\bigwedge_{i=1}^{n} F_i) \to G$ is valid.
3. $(\bigwedge_{i=1}^{n} F_i) \wedge \neg G$ is unsatisfiable.

2.3 Substitution

Substitution A substitution $\sigma : \mathcal{V} \to \mathcal{T}$ is a mapping from variables to terms. It is written as $\{X_1 \mapsto t_1, \dots, X_n \mapsto t_n\}$. Substitution

The application of a substitution is the following:

- $p(t_1, \dots, t_n)\sigma = p(t_1\sigma, \dots, t_n\sigma)$
- $f(t_1, \dots, t_n)\sigma = fp(t_1\sigma, \dots, t_n\sigma)$
- $\bot\sigma = \bot$ and $\top\sigma = \top$
- $(\neg F)\sigma = (\neg F\sigma)$
- $(F_1 \star F_2)\sigma = (F_1\sigma \star F_2\sigma)$ for $\star \in \{\wedge, \vee, \to\}$

- $(\forall X.F)\sigma = \forall X'(F\sigma[X \mapsto X'])$ where X' is a fresh variable (i.e. does not appear in F).
- $(\exists X.F)\sigma = \exists X'(F\sigma[X \mapsto X'])$ where X' is a fresh variable.

Unifier A substitution σ is a unifier for $e_1, \ldots, e_n$ if $e_1\sigma = \cdots = e_n\sigma$. Unifier

Most general unifier A unifier σ is the most general unifier (MGU) for $\bar{e} = e_1, \ldots, e_n$ if every unifier τ for $\bar{e}$ is an instance of σ ($\tau = \sigma\rho$ for some substitution ρ). In other words, σ is the smallest substitution to unify $\bar{e}$. Most general unifier

3 Logic programming

3.1 Syntax

A logic program has the following components (defined using BNF):

Atom $A := p(t_1, \dots, t_n)$ for $n \geq 0$ Atom

Goal $G := \top \mid \bot \mid A \mid G_1 \wedge G_2$ Goal

Horn clause A clause with at most one positive literal. Horn clause

$$K := A \leftarrow G$$

In other words, A and all the literals in G are positive as $A \leftarrow G = A \vee \neg G$.

Program $P := K_1 \dots K_m$ for $m \geq 0$ Program

3.2 Semantics

3.2.1 State transition system

State Pair $\langle G, \theta \rangle$ where G is a goal and θ is a substitution. State

Intial state $\langle G, \varepsilon \rangle$

Successful final state $\langle \top, \theta \rangle$

Failed final state $\langle \bot, \varepsilon \rangle$

Derivation A sequence of states. A derivation can be: Derivation

Successful If the final state is successful.

Failed If the final state is failed.

Infinite If there is an infinite sequence of states.

Given a derivation, a goal G can be:

Successful There is a successful derivation starting from $\langle G, \varepsilon \rangle$.

Finitely failed All the derivations starting from $\langle G, \varepsilon \rangle$ are failed.

Computed answer substitution Given a goal G and a program P, if there exists a successful derivation $\langle G, \varepsilon \rangle \mapsto * \langle \top, \theta \rangle$, then the substitution θ is the computed answer substitution of G. Computed answer substitution

Transition Starting from the state $\langle A \wedge G, \theta \rangle$ of a program P, a transition on the atom A can result in: Transition

Unfold If there exists a clause $(B \leftarrow H)$ in P and a (most general) unifier β for $A\theta$ and B, then we have a transition: $\langle A \wedge G, \theta \rangle \mapsto \langle H \wedge G, \theta\beta \rangle$.

In other words, we want to prove that $A\theta$ holds. To do this, we search for a clause that has as conclusion $A\theta$ and add its premise to the things to prove. If a unification is needed to match $A\theta$, we add it to the substitutions of the state.

Failure If there are no clauses $(B \leftarrow H)$ in P with a unifier for $A\theta$ and B, then we have a transition: $\langle A \wedge G, \theta \rangle \mapsto \langle \bot, \varepsilon \rangle$.

Non-determinism A transition has two types of non-determinism:

Don't-care Any atom in $(A \wedge G)$ can be chosen to determine the next state. This affects the length of the derivation (infinite in the worst case). Don't-care

Don't-know Any clause $(B \rightarrow H)$ in P with an unifier for $A\theta$ and B can be chosen. This determines the output of the derivation. Don't-know

Selective linear definite resolution Approach to avoid non-determinism when constructing a derivation. SLD resolution

Selection rule Method to select the atom in the goal to unfold (eliminates don't-care non-determinism). Selection rule

SLD tree Search tree constructed using all the possible clauses according to a selection rule and visited following a search strategy (eliminates don't know non-determinism). SLD tree

Theorem 3.2.1 (Soundness)**.** Given a program P, a goal G and a substitution θ, if θ is a computed answer substitution, then $P \models G\theta$.

Theorem 3.2.2 (Completeness)**.** Given a program P, a goal G and a substitution θ, if $P \models G\theta$, then it exists a computed answer substitution σ such that $G\theta = G\sigma\beta$.

Theorem 3.2.3. If a computed answer substitution can be obtained using a selection rule r, it can be obtained also using a different selection rule r'.

Prolog SLD Prolog SLD

Selection rule Select the leftmost atom.

Tree search strategy Search following the order of definition of the clauses.

This results in a left-to-right, depth-first search of the SLD tree. Note that this may end up in a loop.

4 Prolog

It may be useful to first have a look at the "Logic programming" section of `Languages and Algorithms for AI (module 2)`.

4.1 Syntax

Term Following the first-order logic definition, a term can be a: Term

- Constant (`lowerCase`).
- Variable (`UpperCase`).
- Function symbol (`f(t1, ..., tn)` with `t1`, ..., `tn` terms).

Atomic formula An atomic formula has form: Atomic formula

`p(t1, ..., tn)`

where `p` is a predicate symbol and `t1`, ..., `tn` are terms.

Note: there are no syntactic distinctions between constants, functions and predicates.

Clause A Prolog program is a set of horn clauses: Horn clause

Fact `A.`

Rule `A :- B1, ..., Bn.` (`A` is the head and `B1, ..., Bn` the body)

Goal `:- B1, ..., Bn.`

where:

- `A`, `B1`, ..., `Bn` are atomic formulas.
- `,` represents the conjunction ($\land$).
- `:-` represents the logical implication ($\Leftarrow$).

Quantification Quantification

Facts Variables appearing in a fact are quantified universally.

$$\texttt{A(X).} \equiv \forall \texttt{X} : \texttt{A(X)}$$

Rules Variables appearing the the body only are quantified existentially. Variables appearing in both the head and the body are quantified universally.

$$\texttt{A(X) :- B(X, Y).} \equiv \forall \texttt{X}, \exists \texttt{Y} : \texttt{A(X)} \Leftarrow \texttt{B(X, Y)}$$

Goals Variables are quantified existentially.

$$\texttt{:- B(Y).} \equiv \exists \texttt{Y} : \texttt{B(Y)}$$

4.2 Semantics

Execution of a program A computation in Prolog attempts to prove the goal. Given a program P and a goal `:- p(t1, ..., tn)`, the objective is to find a substitution σ such that:

$$P \models [\,\texttt{p(t1, ..., tn)}\,]\sigma$$

In practice, it uses two stacks:

Execution stack Contains the predicates the interpreter is trying to prove.

Backtracking stack Contains the choice points (clauses) the interpreter can try.

SLD resolution Prolog uses SLD resolution with the following choices: SLD

Left-most Always proves the left-most literal first.

Depth-first Applies the predicates following the order of definition.

Note that the depth-first approach can be efficiently implemented (tail recursion) but the termination of a Prolog program on a provable goal is not guaranteed as it may loop depending on the ordering of the clauses.

Disjunction operator The operator `;` can be seen as a disjunction and makes the Prolog interpreter explore the remaining SLD tree looking for alternative solutions.

4.3 Arithmetic operators

In Prolog: Arithmetic operators

- Integers and floating points are built-in atoms.
- Math operators are built-in function symbols.

Therefore, mathematical expressions are terms.

`is` predicate The predicate `is` is used to evaluate and unify expressions:

```
T is Expr
```

where `T` is a numerical atom or a variable and `Expr` is an expression without free variables. After evaluation, the result of `Expr` is unified with `T`.

Example.

```
?- X is 2+3.
    yes X=5
```

Note: a term representing an expression is evaluated only with the predicate `is` (otherwise it remains as is).

Relational operators Relational operators (`>`, `<`, `>=`, `=<`, `==`, `=/=`) are built-in.

4.4 Lists

A list is defined recursively as: Lists

Empty list `[]`

List constructor `.(T, L)` where `T` is a term and `L` is a list.

Note that a list always ends with an empty list.
As the formal definition is impractical, some syntactic sugar has been defined:

List definition `[t1, ..., tn]` can be used to define a list.

Head and tail `[H | T]` where `H` is the head (term) and `T` the tail (list) can be useful for recursive calls.

4.5 Cut

The cut operator (`!`) allows to control the exploration of the SLD tree. Cut
A cut in a clause:

```
p :- q1, ..., qi, !, qj, ..., qn.
```

makes the interpreter consider only the first choice points for `q1, ..., qi`, dropping all the other possibilities. Therefore, if `qj, ..., qn` fails, there won't be backtracking and `p` fails.

Example.

```
p(X) :- q(X), r(X).
q(1).
q(2).
r(2).

?- p(X).
    yes X=2
```

```
p(X) :- q(X), !, r(X).
q(1).
q(2).
r(2).

?- p(X).
    no
```

In the second case, the cut drops the choice point `q(2)` and only considers `q(1)`.

Mutual exclusion A cut can be useful to achieve mutual exclusion. In other words, to represent a conditional branching:

```
if a(X) then b else c
```

a cut can be used as follows:

```
p(X) :- a(X), !, b.
p(X) :- c.
```

If `a(X)` succeeds, other choice points for `p` will be dropped and only `b` will be evaluated. If `a(X)` fails, the second clause will be considered, therefore evaluating `c`.

4.6 Negation

Closed-world assumption Only what is stated in a program P is true, everything else is false: Closed-world assumption

$$\texttt{CWA}(P) = P \cup \{\neg A \mid A \text{ is a ground atomic formula and } P \not\models A\}$$

Non-monotonic inference rule Adding new axioms to the program may change the set of valid theorems.

As first-order logic in undecidable, closed-world assumption cannot be directly applied in practice.

Negation as failure A negated atom $\neg A$ is considered true iff A fails in finite time: Negation as failure

$$\texttt{NF}(P) = P \cup \{\neg A \mid A \in \texttt{FF}(P)\}$$

where $\texttt{FF}(P) = \{B \mid P \not\models B \text{ in finite time}\}$ is the set of atoms for which the proof fails in finite time. Note that not all atoms B such that $P \not\models B$ are in $\texttt{FF}(P)$.

SLDNF SLD resolution with NF to solve negative atoms. SLDNF

Given a goal of literals `:- L`$_1$`, ..., L`$_m$, SLDNF does the following:

1. Select a positive or ground negative literal $\texttt{L}_i$:
   - If $\texttt{L}_i$ is positive, apply the normal SLD resolution.
   - If $\texttt{L}_i = \neg A$, prove that A fails in finite time. If it succeeds, $\texttt{L}_i$ fails.
2. Solve the goal `:- L`$_1$`, ..., L`$_{i-1}$`, L`$_{i+1}$`, ...L`$_m$.

Theorem 4.6.1. If only positive or ground negative literal are selected during resolution, SLDNF is correct and complete.

Prolog SLDNF Prolog uses an incorrect implementation of SLDNF where the selection rule always chooses the left-most literal. This potentially causes incorrect deductions.

Proof. When proving `:- \+capital(X).`, the intended meaning is:

$$\exists \texttt{X} : \neg\texttt{capital(X)}$$

In SLDNF, to prove `:- \+capital(X).`, the algorithm proves `:- capital(X).`, which results in:

$$\exists \texttt{X} : \texttt{capital(X)}$$

and then negates the result, which corresponds to:

$$\neg(\exists \texttt{X} : \texttt{capital(X)}) \iff \forall \texttt{X} : (\neg\texttt{capital(X)})$$

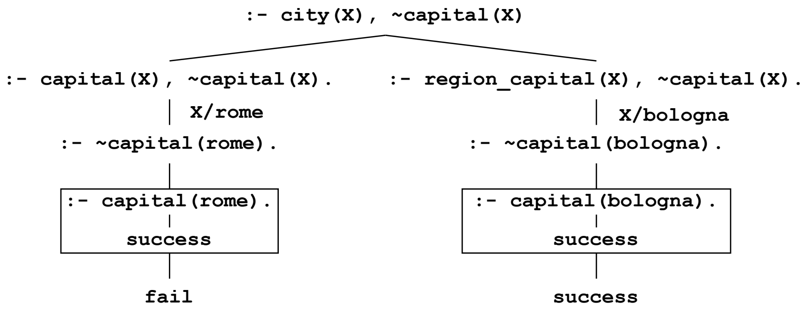
□

Example (Correct SLDNF resolution)**.** Given the program:

```
capital(rome).
region_capital(bologna).
city(X) :- capital(X).
city(X) :- region_capital(X).

?- city(X), \+capital(X).
```

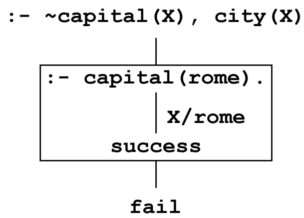
its resolution succeeds with `X=bologna` as `\+capital(X)` is ground by the unification of `city(X)`.



Example (Incorrect SLDNF resolution)**.** Given the program:

```
capital(rome).
region_capital(bologna).
city(X) :- capital(X).
city(X) :- region_capital(X).

?- \+capital(X), city(X).
```



its resolution fails as `\+capital(X)` is a free variable and the proof of `capital(X)` is ground with `X=rome` and succeeds, therefore failing `\+capital(X)`. Note that `bologna` is not tried as it does not appear in the axioms of `capital`.

4.7 Meta predicates

`call/1` Given a term `T`, `call(T)` considers `T` as a predicate and evaluates it. At the time of evaluation, `T` must be a non-numeric term. `call/1`

Example.

```
p(X) :- call(X).
q(a).

?- p(q(Y)).
    yes Y=a
```

`fail/0` The evaluation of `fail` always fails, forcing the interpreter to backtrack. `fail/0`

Example (Implementation of negation as failure)**.**

```
not(P) :- call(P), !, fail.
not(P).
```

Note that the cut followed by `fail` (`!, fail`) is useful to force a global failure.

`bagof/3` and `setof/3`

`bagof/3` The predicate `bagof(X, P, L)` unifies `L` with a list of the instances of `X` that satisfy `P`. Fails if none exists. `bagof/3`

`sefof/3` The predicate `setof(X, P, S)` unifies `S` with a set of the instances of `X` that satisfy `P`. Fails if none exists. `sefof/3`

In practice, for computational reasons, a list (with repetitions) might be computed.

Example.

```
p(1).
p(2).
p(1).

?- setof(X, p(X), S).
    yes S=[1, 2] X=X

?- bagof(X, p(X), S).
    yes S=[1, 2, 1] X=X
```

Quantification When solving a goal, the interpreter unifies free variables with a value. This may cause unwanted behaviors when using `bagof` or `setof`. The `X^` tells the interpreter to not (permanently) bind the variable `X`.

Example.

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).

?- setof(X, father(X, Y), S).
    yes X=X Y=giuseppe S=[giovanni];
        X=X Y=mario    S=[giovanni];
        X=X Y=paola    S=[mario]
```

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).

?- setof(X, Y^father(X, Y), S).
    yes S=[giovanni, mario] X=X Y=Y
```

`findall/3` The predicate `findall(X, P, S)` unifies `S` with a list of the instances of `X` that satisfy `P`. If none exists, `S` is unified with an empty list. Variables in `P` that do not appear in `X` are not bound (same as the `Y^` operator). findall/3

Example.

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).

?- findall(X, father(X, Y), S).
    yes S=[giovanni, mario] X=X Y=Y
```

`var/1` The predicate `var(T)` is true if `T` is a variable. var/1

`nonvar/1` The predicate `nonvar(T)` is true if `T` is not a free variable. nonvar/1

`number/1` The predicate `number(T)` is true if `T` is a number. number/1

`ground/1` The predicate `ground(T)` is true if `T` does not have free variables. ground/1

`=../2` The operator `T =.. L` unifies `L` with a list where its head is the head of `T` and the tail contains the remaining arguments of `T` (i.e. puts all the components of a predicate into a list). Only one between `T` and `L` may be a variable. =../2

Example.

```
?- foo(hello, X) =.. List.
    List = [foo, hello, X]
```

```
?- Term =.. [baz, foo(1)].
    Term = baz(foo(1))
```

`clause/2` The predicate `clause(Head, Body)` is true if it can unify `Head` and `Body` with an existing clause. `Head` must be initialized to a non-numeric term. `Body` can be a variable or a term. clause/2

Example.

```
p(1).
q(X, a) :- p(X), r(a).
q(2, Y) :- d(Y).

?- clause(p(1), B).
    yes B=true

?- clause(p(X), true).
    yes X=1

?- clause(q(X, Y), B).
    yes X=_1 Y=a B=p(_1), r(a);
        X=2 Y=_2 B=d(_2)
```

`assert/1` The predicate `assert(T)` adds `T` in an unspecified position of the clauses database of Prolog. In other words, it allows to dynamically add clauses. assert/1

`asserta/1` As `assert(T)`, with insertion at the beginning of the database. asserta/1

`assertz/1` As `assert(T)`, with insertion at the end of the database. assertz/1

Note that `:- assert((p(X)))` quantifies `X` existentially as it is a query. If it is not ground and added to the database as is, is becomes a clause and therefore quantified universally: $\forall$`X` : `p(X)`.

Example (Lemma generation)**.**

```
fib(0, 0) :- !.
fib(1, 1) :- !.
fib(N, F) :- N1 is N-1, fib(N1, F1),
             N2 is N-2, fib(N2, F2),
             F is F1+F2,
             generate_lemma(fib(N, F)).

generate_lemma(T) :- clause(T, true), !.
generate_lemma(T) :- assert(T).
```

`generate_lemma/1` allows to add to the clauses database all the intermediate steps to compute the Fibonacci sequence (similar concept to dynamic programming).

`retract/1` The predicate `retract(T)` removes from the database the first clause that unifies with `T`. retract/1

`abolish/2` The predicate `abolish(T, n)` removes from the database all the occurrences of `T` with arity `n`. abolish/2

4.8 Meta-interpreters

Meta-interpreter Interpreter for a language L_1 written in another language L_2. Meta-interpreter

Prolog vanilla meta-interpreter The Prolog vanilla meta-interpreter is defined as follows: Vanilla meta-interpreter

```
solve(true) :- !.
solve( (A, B) ) :- !, solve(A), solve(B).
solve(A) :- clause(A, B), solve(B).
```

In other words, the clauses state the following:

1. A tautology is a success.
2. To prove a conjunction, we have to prove both atoms.
3. To prove an atom `A`, we look for a clause `A :- B` that has `A` as conclusion and prove its premise `B`.

5 Constraint programming

Class of problems

Constraint satisfaction problem (CSP) Defined by: Constraint satisfaction problem

- A finite set of variables $X_1, \dots, X_n$.
- A domain for each variable $D(X_1), \dots, D(X_n)$.
- A set of constraints $\{C_1, \dots, C_m\}$

A solution is an assignment to all the variables while satisfying the constraints.

Constraint optimization problem (COP) Extension of a constraint satisfaction problem with an objective function with domain D: Constraint optimization problem

$$f : D(X_1) \times \dots \times D(X_n) \to D$$

A solution is a CSP solution that optimizes f.

Class of languages

Constraint logic programming (CLP) Add constraints and solvers to logic programming. Generally more efficient than plain logic programming. Constraint logic programming

Imperative languages Add constraints and solvers to imperative languages (e.g. MiniZinc). Imperative languages

5.1 Constraint logic programming (CLP)

5.1.1 Syntax

Atom $A := p(t_1, \dots, t_n)$, for $n \geq 0$. p is a predicate. Atom

Constraint $C := c(t_1, \dots, t_n) \mid C_1 \wedge C_2$, for $n \geq 0$. c is an atomic constraint. Constraint

Goal $G := \top \mid \bot \mid A \mid C \mid G_1 \wedge G_2$ Goal

Constraint logic clause $K := A \leftarrow G$ Constraint logic clause

Constraint logic program $P := K_1 \dots K_m$, for $m \geq 0$ Constraint logic program

5.1.2 Semantics

Transition system Transition system

State Pair $\langle G, C \rangle$ where G is a goal and C is a constraint.

Initial state $\langle G, \top \rangle$

Successful final state $\langle \top, C \rangle$ with $C \neq \bot$

Failed final state $\langle G, \bot \rangle$

Transition Starting from the state $\langle A \wedge G, C \rangle$ of a program P, a transition on the atom A can result in:

Unfold If there exists a clause $(B \leftarrow H)$ in P and an assignment $(B \doteq A)$ such that $((B \doteq A) \wedge C)$ is still valid, then we have a transition $\langle A \wedge G, C \rangle \mapsto \langle H \wedge G, (B \doteq A) \wedge C \rangle$. Unfold

In other words, we want to develop an atom A and the current constraints are denoted as C. We look for a clause whose head equals A, applying an assignment if needed. If this is possible, we transit from solving A to solving the body of the clause and add the assignment to the set of active constraints.

Failure If there are no clauses $(B \leftarrow H)$ with a valid assignment $((B \doteq A) \wedge C)$, then we have a transition $\langle A \wedge G, C \rangle \mapsto \langle \perp, \perp \rangle$. Failure

Moreover, starting from the state $\langle C \wedge G, D_1 \rangle$ of a program P, a transition on the constraint C can result in:

Solve If $(C \wedge D_1) \iff D_2$ holds, then we have a transition $\langle C \wedge G, D_1 \rangle \mapsto \langle G, D_2 \rangle$. Solve

In other words, we want to develop a constraint C and the current constraints are denoted as D_1. If $(C \wedge D_1)$ is valid, we call it D_2 and continue solving the rest of the goal constrained to D_2.

Non-determinism As in logic programming, there is don't-care and don't-know non-determinism. A SLD search tree is also used.

Derivation strategies

Generate-and-test Strategy adopted by logic programs. Every possible assignment to the variables are generated and tested. Generate-and-test

The workflow is the following:

1. Determine domain.
2. Make an assignment to each variable.
3. Test the constraints.

Constrain-and-generate Strategy adopted by constraint logic programs. Exploit facts to reduce the search space. Constrain-and-generate

The workflow is the following:

1. Determine domain.
2. Restrict the domain following the constraints.
3. Make an assignment to each variable.