Languages and Algorithms for Artificial Intelligence (Module 2)

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Alma Mater Studiorum · University of Bologna

Contents

1 Propositional logic

1.1 Syntax

Syntax Rules and symbols to define well formed sentences. Syntax

The symbols of propositional logic are:

Proposition symbols p_0, p_1, ...

Connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, $\bot$

Auxiliary symbols (and)

Well-formed formula The definition of a well-formed formula is recursive: Well-formed formula

- An atomic proposition is a well-formed formula.
- If S is well-formed, $\neg S$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \wedge S_2$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \vee S_2$ is well-formed.

Note that the implication $S_1 \rightarrow S_2$ can be written as $\neg S_1 \vee S_2$.

1.1.1 Propositional logic BNF

```
<formula> := atomic_proposition |
             ¬<formula> |
             <formula> ∧ <formula> |
             <formula> ∨ <formula> |
             <formula> → <formula> |
             <formula> ↔ <formula> |
             (<formula>)
```

1.2 Semantics

Semantics Rules to associate a meaning to well formed sentences. Semantics

Model theory What is true.

Proof theory What is provable.

Interpretation Given a propositional formula F of n atoms $\{A_1, \dots, A_n\}$, an interpretation I of F is an assignment of truth values to $\{A_1, \dots, A_n\}$. Interpretation

Note: given a formula F of n distinct atoms, there are 2^n district interpretations.

Model If F is true under the interpretation I, we say that I is a model of F ($I \models F$). Model

Truth table Useful to define the semantics of connectives. Truth table

- $\neg S$ is true iff S is false.
- $S_1 \wedge S_2$ is true iff S_1 is true and S_2 is true.
- $S_1 \vee S_2$ is true iff S_1 is true or S_2 is true.
- $S_1 \rightarrow S_2$ is true iff S_1 is false or S_2 is true.
- $S_1 \leftrightarrow S_2$ is true iff $S_1 \rightarrow S_2$ is true and $S_1 \leftarrow S_2$ is true.

Evaluation order

Evaluation The connectives of a propositional formula are evaluated in the order:

$$\leftrightarrow, \rightarrow, \vee, \wedge, \neg$$

Formulas in parenthesis have higher priority.

Valid formula

Valid formula A formula F is valid (tautology) iff it is true in all the possible interpretations. It is denoted as $\models F$.

Invalid formula

Invalid formula A formula F is invalid iff it is not valid (:o).

In other words, there is at least an interpretation where F is false.

Inconsistent formula

Inconsistent formula A formula F is inconsistent (unsatisfiable) iff it is false in all the possible interpretations.

Consistent formula

Consistent formula A formula F is consistent (satisfiable) iff it is not inconsistent.

In other words, there is at least an interpretation where F is true.

Decidability

Decidability A propositional formula is decidable if there is a terminating method to decide if it is valid.

Logical consequence

Logical consequence Let $\Gamma = \{F_1, \dots, F_n\}$ be a set of formulas and G a formula. G is a logical consequence of Γ ($\Gamma \models G$) if in all the possible interpretations I, if $F_1 \wedge \dots \wedge F_n$ is true, G is true.

Logical equivalence

Logical equivalence Two formulas F and G are logically equivalent ($F \equiv G$) iff the truth values of F and G are the same under the same interpretation. In other words, $F \models G$ and $G \models F$.

Common equivalences are:

Commutativity : $(P \wedge Q) \equiv (Q \wedge P)$ and $(P \vee Q) \equiv (Q \vee P)$

Associativity : $((P \wedge Q) \wedge R) \equiv (P \wedge (Q \wedge R))$ and $((P \vee Q) \vee R) \equiv (P \vee (Q \vee R))$

Double negation elimination : $\neg(\neg P) \equiv P$

Contraposition : $(P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)$

Implication elimination : $(P \rightarrow Q) \equiv (\neg P \vee Q)$

Biconditional elimination : $(P \leftrightarrow Q) \equiv ((P \rightarrow Q) \wedge (Q \rightarrow P))$

De Morgan : $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ and $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Distributivity of $\wedge$ over $\vee$: $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$

Distributivity of $\vee$ over $\wedge$: $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$

Deduction theorem

Theorem 1.2.1 (Deduction)**.** Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G:

$$(F_1 \wedge \dots \wedge F_n) \models G \iff \models (F_1 \wedge \dots \wedge F_n) \rightarrow G$$

Proof.

$\rightarrow$) By hypothesis $(F_1 \wedge \cdots \wedge F_n) \models G$.

So, for each interpretation I in which $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. Therefore, $I \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

Moreover, for each interpretation I' in which $(F_1 \wedge \cdots \wedge F_n)$ is false, $(F_1 \wedge \cdots \wedge F_n) \rightarrow G$ is true. Therefore, $I' \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

In conclusion, $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

$\leftarrow$) By hypothesis $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$. Therefore, for each interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true.

In conclusion, $(F_1 \wedge \cdots \wedge F_n) \models G$.

□

Theorem 1.2.2 (Refutation)**.** Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G: Refutation theorem

$$(F_1 \wedge \cdots \wedge F_n) \models G \iff F_1 \wedge \cdots \wedge F_n \wedge \neg G \text{ is inconsistent}$$

Note: this theorem is not accepted in intuitionistic logic.

Proof. By definition, $(F_1 \wedge \cdots \wedge F_n) \models G$ iff for every interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. This requires that there are no interpretations where $(F_1 \wedge \cdots \wedge F_n)$ is true and G false. In other words, it requires that $(F_1 \wedge \cdots \wedge F_n \wedge \neg G)$ is inconsistent. □

1.2.1 Normal forms

Negation normal form (NNF) A formula is in negation normal form iff negations appears only in front of atoms (i.e. not parenthesis). Negation normal form

Conjunctive normal form (CNF) A formula F is in conjunctive normal form iff Conjunctive normal form

- it is in negation normal form;
- it has the form $F := F_1 \wedge F_2 \cdots \wedge F_n$, where each F_i (clause) is a disjunction of literals.

Example.
$(\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF.
$\neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF (not in NNF).

Disjunctive normal form (DNF) A formula F is in disjunctive normal form iff Disjunctive normal form

- it is in negation normal form;
- it has the form $F := F_1 \vee F_2 \cdots \vee F_n$, where each F_i is a conjunction of literals.

1.3 Natural deduction

Proof theory Set of rules that allows to derive conclusions from premises by exploiting syntactic manipulations. Proof theory

Natural deduction Set of rules to introduce or eliminate connectives. We consider a subset $\{\wedge, \rightarrow, \bot\}$ of functionally complete connectives. Natural deduction for propositional logic

Natural deduction can be represented using a tree like structure:

$$\cfrac{\begin{matrix}[\text{hypothesis}]\\ \vdots \\ \text{premise}\end{matrix}}{\text{conclusion}}\ \text{rule name}$$

The conclusion is true when the hypothesis are able to prove the premise. Another tree can be built on top of premises to prove them.

Introduction Usually used to prove the conclusion by splitting it.

Introduction rules

$$\frac{\psi \qquad \varphi}{\varphi \land \psi}\ \land\mathrm{I} \qquad\qquad \cfrac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\ {\to}\mathrm{I}$$

Elimination Usually used to exploit hypothesis and derive a conclusion.

Elimination rules

$$\frac{\varphi \land \psi}{\varphi}\ \land\mathrm{E} \qquad \frac{\varphi \land \psi}{\psi}\ \land\mathrm{E} \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}\ {\to}\mathrm{E}$$

Ex falso sequitur quodlibet From contradiction, anything follows. This can be used when we have two contradicting hypothesis.

Ex falso sequitur quodlibet

$$\frac{\bot}{\varphi}\ \bot$$

Reductio ad absurdum Assume the opposite and prove a contradiction (not accepted in intuitionistic logic).

Reductio ad absurdum

$$\cfrac{\begin{matrix}[\neg\varphi]\\ \vdots \\ \bot\end{matrix}}{\varphi}\ \mathrm{RAA}$$