Fundamentals of Artificial Intelligence and Knowledge Representation (Module 3)

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Alma Mater Studiorum · University of Bologna

Contents

1 Introduction

1.1 Uncertainty

Uncertainty A task is uncertain if we have: Uncertainty

- Partial observations
- Noisy or wrong information
- Uncertain action outcomes
- Complex models

A purely logic approach leads to:

- Risks falsehood: unreasonable conclusion when applied in practice.
- Weak decisions: too many conditions required to make a conclusion.

1.1.1 Handling uncertainty

Default/nonmonotonic logic Works on assumptions. An assumption can be contradicted by an evidence. Default/nonmonotonic logic

Rule-based systems with fudge factors Formulated as premise $\rightarrow_{\text{prob.}}$ effect. Have the following issues: Rule-based systems with fudge factors

- Locality: how can the probability account all the evidence.
- Combination: chaining of unrelated concepts.

Probability Assign a probability given the available known evidence. Probability

Note: fuzzy logic handles the degree of truth and not the uncertainty.

Decision theory Defined as: Decision theory

$$\text{Decision theory} = \text{Utility theory} + \text{Probability theory}$$

where the utility theory depends on one's preferences.

2 Probability

Sample space Set Ω of all possible worlds. Sample space

Event Subset $A \subseteq \Omega$. Event

Sample point/Possible world/Atomic event Element $\omega \in \Omega$. Sample point

Probability space A probability space/model is a function $\mathcal{P}(\cdot) : \Omega \to [0, 1]$ assigned to a sample space such that: Probability space

- $0 \leq \mathcal{P}(\omega) \leq 1$
- $\sum_{\omega \in \Omega} \mathcal{P}(\omega) = 1$
- $\mathcal{P}(A) = \sum_{\omega \in A} \mathcal{P}(\omega)$

Random variable A function from an event to some range (e.g. reals, booleans, ...). Random variable

Probability distribution For any random variable X: Probability distribution

$$\mathcal{P}(X = x_i) = \sum_{\omega \text{ st } X(\omega)=x_i} \mathcal{P}(\omega)$$

Proposition Event where a random variable has a certain value. Proposition

$$a = \{\omega \mid A(\omega) = \texttt{true}\}$$

$$\neg a = \{\omega \mid A(\omega) = \texttt{false}\}$$

$$(\texttt{Weather} = \texttt{rain}) = \{\omega \mid B(\omega) = \texttt{rain}\}$$

Prior probability Prior/unconditional probability of a proposition based on known evidence. Prior probability

Probability distribution (all) Gives all the probabilities of a random variable. Probability distribution (all)

$$\mathbf{P}(A) = \langle \mathcal{P}(A = a_1), \dots, \mathcal{P}(A = a_n) \rangle$$

Joint probability distribution The joint probability distribution of a set of random variables gives the probability of all the different combinations of their atomic events. Joint probability distribution

Note: Every question on a domain can, in theory, be answered using the joint distribution. In practice, it is hard to apply.

Example. $\mathbf{P}(\texttt{Weather}, \texttt{Cavity}) =$

	Weather=sunny	Weather=rain	Weather=cloudy	Weather=snow
Cavity=true	0.144	0.02	0.016	0.02
Cavity=false	0.576	0.08	0.064	0.08

Probability density function The probability density function (PDF) of a random variable X is a function $p : \mathbb{R} \to \mathbb{R}$ such that: Probability density function

$$\int_{\mathcal{T}_X} p(x) \, dx = 1$$

Uniform distribution Uniform distribution

$$p(x) = \text{Unif}[a,b](x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Gaussian (normal) distribution Gaussian (normal) distribution

$$\mathcal{N}(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$\mathcal{N}(0,1)$ is the standard Gaussian.

Conditional probability Probability of a prior knowledge with new evidence: Conditional probability

$$\mathcal{P}(a|b) = \frac{\mathcal{P}(a \land b)}{\mathcal{P}(b)}$$

The product rule gives an alternative formulation:

$$\mathcal{P}(a \land b) = \mathcal{P}(a|b)\mathcal{P}(b) = \mathcal{P}(b|a)\mathcal{P}(a)$$

Chain rule Successive application of the product rule: Chain rule

$$\begin{aligned}
\mathbf{P}(X_1, \dots, X_n) &= \mathbf{P}(X_1, \dots, X_{n-1})\mathbf{P}(X_n | X_1, \dots, X_{n-1}) \\
&= \mathbf{P}(X_1, \dots, X_{n-2})\mathbf{P}(X_{n-1} | X_1, \dots, X_{n-2})\mathbf{P}(X_n | X_1, \dots, X_{n-1}) \\
&= \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, \dots, X_{i-1})
\end{aligned}$$

Independence Two random variables A and B are independent ($A \perp B$) iff: Independence

$$\mathbf{P}(A|B) = \mathbf{P}(A) \text{ or } \mathbf{P}(B|A) = \mathbf{P}(B) \text{ or } \mathbf{P}(A,B) = \mathbf{P}(A)\mathbf{P}(B)$$

Conditional independence Two random variables A and B are conditionally independent iff: Conditional independence

$$\mathbf{P}(A \,|\, C, B) = \mathbf{P}(A \,|\, C)$$

2.1 Inference with full joint distributions

Given a joint distribution, the probability of any proposition ϕ can be computed as the sum of the atomic events where ϕ is true:

$$\mathcal{P}(\phi) = \sum_{\omega : \omega \models \phi} \mathcal{P}(\omega)$$

Example. Given the following joint distribution:

	toothache		¬toothache	
	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

We have that:

- $\mathcal{P}(\texttt{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$
- $\mathcal{P}(\texttt{cavity} \vee \texttt{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$
- $\mathcal{P}(\neg\texttt{cavity} \,|\, \texttt{toothache}) = \frac{\mathcal{P}(\neg\texttt{cavity} \wedge \texttt{toothache})}{\mathcal{P}(\texttt{toothache})} = \frac{0.016+0.064}{0.2} = 0.4$

Marginalization The probability that a random variable assumes a specific value is given by the sum off all the joint probabilities where that random variable assumes the given value.

Marginalization

Example. Given the joint distribution:

	Weather=sunny	Weather=rain	Weather=cloudy	Weather=snow
Cavity=true	0.144	0.02	0.016	0.02
Cavity=false	0.576	0.08	0.064	0.08

We have that $\mathcal{P}(\texttt{Weather} = \texttt{sunny}) = 0.144 + 0.576$

Conditioning Adding a condition to a probability (reduction and renormalization).

Conditioning

Normalization Given a conditional probability distribution $\mathbf{P}(A|B)$, it can be formulated as:

Normalization

$$\mathbf{P}(A|B) = \alpha\mathbf{P}(A, B)$$

where α is a normalization constant. In fact, fixed the evidence B, the denominator to compute the conditional probability is the same for each probability.

Example. Given the joint distribution:

	toothache		¬toothache	
	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

We have that:

$$\mathbf{P}(\texttt{cavity}|\texttt{toothache}) = \langle \frac{\mathcal{P}(\texttt{cavity}, \texttt{toothache}, \texttt{catch})}{\mathcal{P}(\texttt{toothache})}, \frac{\mathcal{P}(\texttt{cavity}, \texttt{toothache}, \neg\texttt{catch})}{\mathcal{P}(\texttt{toothache})} \rangle$$

Probability query Given a set of query variables $\boldsymbol{Y}$, the evidence variables $\mathbf{e}$ and the other hidden variables $\boldsymbol{H}$, the probability of the query can be computed as:

Probability query

$$\mathbf{P}(\boldsymbol{Y}|\boldsymbol{E} = \mathbf{e}) = \alpha\mathbf{P}(\boldsymbol{Y}|\boldsymbol{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\boldsymbol{Y}|\boldsymbol{E} = \mathbf{e}, \boldsymbol{H} = \mathbf{h})$$

The problem of this approach is that it has exponential time and space complexity which makes it not applicable in practice.