Languages and Algorithms for Artificial Intelligence (Module 2)

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Contents

1 Propositional logic

1.1 Syntax

Syntax Rules and symbols to define well-formed sentences. Syntax

The symbols of propositional logic are:

Proposition symbols $p_0, p_1, \dots$

Connectives $\wedge \vee \rightarrow \leftrightarrow \neg \perp ( )$

Well-formed formula The definition of a well-formed formula is recursive: Well-formed formula

- An atomic proposition is a well-formed formula.
- If S is well-formed, $\neg S$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \wedge S_2$ is well-formed.
- If S_1 and S_2 are well-formed, $S_1 \vee S_2$ is well-formed.

Note that the implication $S_1 \rightarrow S_2$ can be written as $\neg S_1 \vee S_2$.

The BNF definition of a formula is:

$$F := \texttt{atomic\_proposition} \,|\, F \wedge F \,|\, F \vee F \,|\, F \rightarrow F \,|\, F \leftrightarrow F \,|\, \neg F \,|\, (F)$$

1.2 Semantics

Semantics Rules to associate a meaning to well-formed sentences. Semantics

Model theory What is true.

Proof theory What is provable.

Interpretation Given a propositional formula F of n atoms $\{A_1, \dots, A_n\}$, an interpretation $\mathcal{I}$ of F is is a pair (D, I) where: Interpretation

- D is the domain. Truth values in the case of propositional logic.
- I is the interpretation mapping that assigns to the atoms $\{A_1, \dots, A_n\}$ an element of D.

Note: given a formula F of n distinct atoms, there are 2^n district interpretations.

Model If F is true under the interpretation $\mathcal{I}$, we say that $\mathcal{I}$ is a model of F ($\mathcal{I} \models F$). Model

Valid formula A formula F is valid (tautology) iff it is true in all the possible interpretations. It is denoted as $\models F$. Valid formula

Invalid formula A formula F is invalid iff it is not valid (:o). Invalid formula

In other words, there is at least an interpretation where F is false.

Inconsistent formula A formula F is inconsistent (unsatisfiable) iff it is false in all the possible interpretations. Inconsistent formula

Consistent formula A formula F is consistent (satisfiable) iff it is not inconsistent. Consistent formula

In other words, there is at least an interpretation where F is true.

Decidability A logic is decidable if there is a terminating method to decide if a formula is valid. Decidability

Propositional logic is decidable.

Truth table Useful to define the semantics of connectives. Truth table

- $\neg S$ is true iff S is false.
- $S_1 \wedge S_2$ is true iff S_1 is true and S_2 is true.
- $S_1 \vee S_2$ is true iff S_1 is true or S_2 is true.
- $S_1 \rightarrow S_2$ is true iff S_1 is false or S_2 is true.
- $S_1 \leftrightarrow S_2$ is true iff $S_1 \rightarrow S_2$ is true and $S_1 \leftarrow S_2$ is true.

Evaluation The connectives of a propositional formula are evaluated in the order: Evaluation order

$$\leftrightarrow, \rightarrow, \vee, \wedge, \neg$$

Formulas in parenthesis have higher priority.

Logical consequence Let $\Gamma = \{F_1, \dots, F_n\}$ be a set of formulas (premises) and G a formula (conclusion). G is a logical consequence of Γ ($\Gamma \models G$) if in all the possible interpretations $\mathcal{I}$, if $F_1 \wedge \dots \wedge F_n$ is true, G is true. Logical consequence

Logical equivalence Two formulas F and G are logically equivalent ($F \equiv G$) iff the truth values of F and G are the same under the same interpretation. In other words, $F \equiv G \iff F \models G \wedge G \models F$. Logical equivalence

Common equivalences are:

Commutativity : $(P \wedge Q) \equiv (Q \wedge P)$ and $(P \vee Q) \equiv (Q \vee P)$

Associativity : $((P \wedge Q) \wedge R) \equiv (P \wedge (Q \wedge R))$ and $((P \vee Q) \vee R) \equiv (P \vee (Q \vee R))$

Double negation elimination : $\neg(\neg P) \equiv P$

Contraposition : $(P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)$

Implication elimination : $(P \rightarrow Q) \equiv (\neg P \vee Q)$

Biconditional elimination : $(P \leftrightarrow Q) \equiv ((P \rightarrow Q) \wedge (Q \rightarrow P))$

De Morgan : $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ and $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

Distributivity of $\wedge$ over $\vee$: $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$

Distributivity of $\vee$ over $\wedge$: $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$

1.2.1 Normal forms

Negation normal form (NNF) A formula is in negation normal form iff negations appear only in front of atoms (i.e. not parenthesis). Negation normal form

Conjunctive normal form (CNF) A formula F is in conjunctive normal form iff: Conjunctive normal form

- it is in negation normal form;

- it has the form $F := F_1 \wedge F_2 \cdots \wedge F_n$, where each F_i (clause) is a disjunction of literals.

Example.
$(\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF.
$\neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF (not in NNF).

Disjunctive normal form (DNF) A formula F is in disjunctive normal form iff: Disjunctive normal form

- it is in negation normal form;
- it has the form $F := F_1 \vee F_2 \cdots \vee F_n$, where each F_i is a conjunction of literals.

1.3 Reasoning

Reasoning method Systems to work with symbols. Reasoning method

Given a set of formulas Γ, a formula F and a reasoning method E, we denote with $\Gamma \vdash^E F$ the fact that F can be deduced from Γ using the reasoning method E.

Sound A reasoning method E is sound iff: Soundness

$$(\Gamma \vdash^E F) \rightarrow (\Gamma \models F)$$

Complete A reasoning method E is complete iff: Completeness

$$(\Gamma \models F) \rightarrow (\Gamma \vdash^E F)$$

Deduction theorem Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G: Deduction theorem

$$(F_1 \wedge \cdots \wedge F_n) \models G \iff \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$$

Proof.

$\rightarrow$) By hypothesis $(F_1 \wedge \cdots \wedge F_n) \models G$.

So, for each interpretation $\mathcal{I}$ in which $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. Therefore, $\mathcal{I} \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

Moreover, for each interpretation $\mathcal{I}'$ in which $(F_1 \wedge \cdots \wedge F_n)$ is false, $(F_1 \wedge \cdots \wedge F_n) \rightarrow G$ is true. Therefore, $\mathcal{I}' \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

In conclusion, $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

$\leftarrow$) By hypothesis $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$. Therefore, for each interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true.

In conclusion, $(F_1 \wedge \cdots \wedge F_n) \models G$.

$\square$

Refutation theorem Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula G: Refutation theorem

$$(F_1 \wedge \cdots \wedge F_n) \models G \iff F_1 \wedge \cdots \wedge F_n \wedge \neg G \text{ is inconsistent}$$

Note: this theorem is not accepted in intuitionistic logic.

Proof. By definition, $(F_1 \wedge \cdots \wedge F_n) \models G$ iff for every interpretation where $(F_1 \wedge \cdots \wedge F_n)$ is true, G is also true. This requires that there are no interpretations where $(F_1 \wedge \cdots \wedge F_n)$ is true and G false. In other words, it requires that $(F_1 \wedge \cdots \wedge F_n \wedge \neg G)$ is inconsistent. $\square$

1.3.1 Natural deduction

Proof theory Set of rules that allows to derive conclusions from premises by exploiting syntactic manipulations. Proof theory

Natural deduction Set of rules to introduce or eliminate connectives. We consider a subset $\{\wedge, \rightarrow, \bot\}$ of functionally complete connectives. Natural deduction for propositional logic

Natural deduction can be represented using a tree like structure:

$$\dfrac{\begin{matrix}[\text{hypothesis}] \\ \vdots \\ \text{premise}\end{matrix}}{\text{conclusion}}\ \text{rule name}$$

The conclusion is true when the hypothesis are able to prove the premise. Another tree can be built on top of premises to prove them.

Introduction Usually used to prove the conclusion by splitting it. Introduction rules

$$\dfrac{\psi \qquad \varphi}{\varphi \wedge \psi}\ \wedge\text{I} \qquad\qquad \dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi}\ \rightarrow\text{I}$$

Elimination Usually used to exploit hypothesis and derive a conclusion. Elimination rules

$$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{E} \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{E} \qquad \dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \rightarrow\text{E}$$

Ex falso sequitur quodlibet From contradiction, anything follows. This can be used when we have two contradicting hypothesis. Ex falso sequitur quodlibet

$$\dfrac{\bot}{\varphi}\ \bot$$

Reductio ad absurdum Assume the opposite and prove a contradiction (not accepted in intuitionistic logic). Reductio ad absurdum

$$\dfrac{\begin{matrix}[\neg\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi}\ \text{RAA}$$

2 First order logic

2.1 Syntax

The symbols of propositional logic are: Syntax

Constants Known elements of the domain. Do not represent truth values.

Variables Unknown elements of the domain. Do not represent truth values.

Function symbols Function $f^{(n)}$ applied on n constants to obtain another constant.

Predicate symbols Function $P^{(n)}$ applied on n constants to obtain a truth value.

Connectives $\forall$ $\exists$ $\wedge$ $\vee$ $\rightarrow$ $\neg$ $\leftrightarrow$ $\top$ $\bot$ ( )

Using the basic syntax, the following constructs can be defined:

Term Denotes elements of the domain.

$$t := \texttt{constant} \mid \texttt{variable} \mid f^{(n)}(t_1, \dots, t_n)$$

Proposition Denotes truth values.

$$P := \top \mid \bot \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid P \leftrightarrow P \mid \neg P \mid \forall x.P \mid \exists x.P \mid (P) \mid P^{(n)}(t_1, \dots, t_n)$$

Well-formed formula The definition of well-formed formula in first order logic extends the one of propositional logic by adding the following conditions: Well-formed formula

- If S is well-formed, $\exists X.S$ is well-formed. Where X is a variable.
- If S is well-formed, $\forall X.S$ is well-formed. Where X is a variable.

Free variables The universal and existential quantifiers bind their variable within the scope of the formula. Let $F_v(F)$ be the set of free variables in a formula F, F_v is defined as follows: Free variables

- $F_v(p(t)) = \bigcup \texttt{vars}(t)$
- $F_v(\top) = F_v(\bot) = \varnothing$
- $F_v(\neg F) = F_v(F)$
- $F_v(F_1 \wedge F_2) = F_v(F_1 \vee F_2) = F_v(F_1 \rightarrow F_2) = F_v(F_1) \cup F_v(F_2)$
- $F_v(\forall X.F) = F_v(\exists X.F) = F_v(F) \smallsetminus \{X\}$

Closed formula/Sentence Proposition without free variables. Sentence

Theory Set of sentences. Theory

Ground term/Formula Proposition without variables. Formula

2.2 Semantics

Interpretation An interpretation in first order logic $\mathcal{I}$ is a pair (D, I): Interpretation

- D is the domain of the terms.
- I is the interpretation function such that:
  - $I(f) : D^n \to D$ for every n-ary function symbol.
  - $I(p) \subseteq D^n$ for every n-ary predicate symbol.

Variable evaluation Given an interpretation $\mathcal{I} = (D, I)$ and a set of variables $\mathcal{V}$, a variable is evaluated through $\eta : \mathcal{V} \to D$. Variable evaluation

Model Given an interpretation $\mathcal{I}$ and a formula F, $\mathcal{I}$ models F ($\mathcal{I} \models F$) when $\mathcal{I}, \eta \models F$ for every variable evaluation η. Model

A sentence S is:

Valid S is satisfied by every interpretation ($\forall \mathcal{I} : \mathcal{I} \models S$).

Satisfiable S is satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \models S$).

Falsifiable S is not satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \not\models S$).

Unsatisfiable S is not satisfied by any interpretation ($\forall \mathcal{I} : \mathcal{I} \not\models S$).

Logical consequence A sentence T_1 is a logical consequence of T_2 ($T_2 \models T_1$) if every model of T_2 is also model of T_1: Logical consequence

$$\mathcal{I} \models T_2 \to \mathcal{I} \models T_1$$

Theorem 2.2.1. It is undecidable to determine if a first order logic formula is a tautology.

Equivalence A sentence T_1 is equivalent to T_2 if $T_1 \models T_2$ and $T_2 \models T_1$. Equivalence

Theorem 2.2.2. The following statements are equivalent:

1. $F_1, \dots, F_n \models G$.
2. $(\bigwedge_{i=1}^{n} F_i) \to G$ is valid.
3. $(\bigwedge_{i=1}^{n} F_i) \land \neg G$ is unsatisfiable.

2.3 Substitution

Substitution A substitution $\sigma : \mathcal{V} \to \mathcal{T}$ is a mapping from variables to terms. It is written as $\{X_1 \mapsto t_1, \dots, X_n \mapsto t_n\}$. Substitution

The application of a substitution is the following:

- $p(t_1, \dots, t_n)\sigma = p(t_1\sigma, \dots, t_n\sigma)$
- $f(t_1, \dots, t_n)\sigma = fp(t_1\sigma, \dots, t_n\sigma)$
- $\bot\sigma = \bot$ and $\top\sigma = \top$
- $(\neg F)\sigma = (\neg F\sigma)$
- $(F_1 \star F_2)\sigma = (F_1\sigma \star F_2\sigma)$ for $\star \in \{\land, \lor, \to\}$

- $(\forall X.F)\sigma = \forall X'(F\sigma[X \mapsto X'])$ where X' is a fresh variable (i.e. does not appear in F).
- $(\exists X.F)\sigma = \exists X'(F\sigma[X \mapsto X'])$ where X' is a fresh variable.

Unifier A substitution σ is a unifier for $e_1, \ldots, e_n$ if $e_1\sigma = \cdots = e_n\sigma$. Unifier

Most general unifier A unifier σ is the most general unifier (MGU) for $\bar{e} = e_1, \ldots, e_n$ if every unifier τ for $\bar{e}$ is an instance of σ ($\tau = \sigma\rho$ for some substitution ρ). In other words, σ is the smallest substitution to unify $\bar{e}$. Most general unifier