# Languages and Algorithms for Artificial Intelligence (Module 2)

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Alma Mater Studiorum · University of Bologna

# Contents

# 1 Propositional logic

## 1.1 Syntax

**Syntax** Rules and symbols to define well-formed sentences. Syntax

The symbols of propositional logic are:

**Proposition symbols** $p_0$, $p_1$, ...

**Connectives** $\wedge \vee \rightarrow \leftrightarrow \neg \perp$ ( )

**Well-formed formula** The definition of a well-formed formula is recursive: Well-formed formula

- An atomic proposition is a well-formed formula.
- If $S$ is well-formed, $\neg S$ is well-formed.
- If $S_1$ and $S_2$ are well-formed, $S_1 \wedge S_2$ is well-formed.
- If $S_1$ and $S_2$ are well-formed, $S_1 \vee S_2$ is well-formed.

Note that the implication $S_1 \rightarrow S_2$ can be written as $\neg S_1 \vee S_2$.

The BNF definition of a formula is:

$$F := \texttt{atomic\_proposition} \,|\, F \wedge F \,|\, F \vee F \,|\, F \rightarrow F \,|\, F \leftrightarrow F \,|\, \neg F \,|\, (F)$$

## 1.2 Semantics

**Semantics** Rules to associate a meaning to well-formed sentences. Semantics

**Model theory** What is true.

**Proof theory** What is provable.

**Interpretation** Given a propositional formula $F$ of $n$ atoms $\{A_1, \dots, A_n\}$, an interpretation $\mathcal{I}$ of $F$ is is a pair $(D, I)$ where: Interpretation

- $D$ is the domain. Truth values in the case of propositional logic.
- $I$ is the interpretation mapping that assigns to the atoms $\{A_1, \dots, A_n\}$ an element of $D$.

Note: given a formula $F$ of $n$ distinct atoms, there are $2^n$ district interpretations.

**Model** If $F$ is true under the interpretation $\mathcal{I}$, we say that $\mathcal{I}$ is a model of $F$ ($\mathcal{I} \models F$). Model

**Valid formula** A formula $F$ is valid (tautology) iff it is true in all the possible interpretations. It is denoted as $\models F$. Valid formula

**Invalid formula** A formula $F$ is invalid iff it is not valid (:o). Invalid formula

In other words, there is at least an interpretation where $F$ is false.

**Inconsistent formula** A formula $F$ is inconsistent (unsatisfiable) iff it is false in all the possible interpretations. Inconsistent formula

**Consistent formula** A formula $F$ is consistent (satisfiable) iff it is not inconsistent. Consistent formula

In other words, there is at least an interpretation where $F$ is true.

**Decidability** A logic is decidable if there is a terminating method to decide if a formula is valid. Decidability

Propositional logic is decidable.

**Truth table** Useful to define the semantics of connectives. Truth table

- $\neg S$ is true iff $S$ is false.
- $S_1 \land S_2$ is true iff $S_1$ is true and $S_2$ is true.
- $S_1 \lor S_2$ is true iff $S_1$ is true or $S_2$ is true.
- $S_1 \to S_2$ is true iff $S_1$ is false or $S_2$ is true.
- $S_1 \leftrightarrow S_2$ is true iff $S_1 \to S_2$ is true and $S_1 \leftarrow S_2$ is true.

**Evaluation** The connectives of a propositional formula are evaluated in the order: Evaluation order

$$\leftrightarrow, \to, \lor, \land, \neg$$

Formulas in parenthesis have higher priority.

**Logical consequence** Let $\Gamma = \{F_1, \dots, F_n\}$ be a set of formulas (premises) and $G$ a formula (conclusion). $G$ is a logical consequence of $\Gamma$ ($\Gamma \models G$) if in all the possible interpretations $\mathcal{I}$, if $F_1 \land \dots \land F_n$ is true, $G$ is true. Logical consequence

**Logical equivalence** Two formulas $F$ and $G$ are logically equivalent ($F \equiv G$) iff the truth values of $F$ and $G$ are the same under the same interpretation. In other words, $F \equiv G \iff F \models G \land G \models F$. Logical equivalence

Common equivalences are:

**Commutativity** : $(P \land Q) \equiv (Q \land P)$ and $(P \lor Q) \equiv (Q \lor P)$

**Associativity** : $((P \land Q) \land R) \equiv (P \land (Q \land R))$ and $((P \lor Q) \lor R) \equiv (P \lor (Q \lor R))$

**Double negation elimination** : $\neg(\neg P) \equiv P$

**Contraposition** : $(P \to Q) \equiv (\neg Q \to \neg P)$

**Implication elimination** : $(P \to Q) \equiv (\neg P \lor Q)$

**Biconditional elimination** : $(P \leftrightarrow Q) \equiv ((P \to Q) \land (Q \to P))$

**De Morgan** : $\neg(P \land Q) \equiv (\neg P \lor \neg Q)$ and $\neg(P \lor Q) \equiv (\neg P \land \neg Q)$

**Distributivity of $\land$ over $\lor$** : $(P \land (Q \lor R)) \equiv ((P \land Q) \lor (P \land R))$

**Distributivity of $\lor$ over $\land$** : $(P \lor (Q \land R)) \equiv ((P \lor Q) \land (P \lor R))$

### 1.2.1 Normal forms

**Negation normal form (NNF)** A formula is in negation normal form iff negations appear only in front of atoms (i.e. not parenthesis). Negation normal form

**Conjunctive normal form (CNF)** A formula $F$ is in conjunctive normal form iff: Conjunctive normal form

- it is in negation normal form;

- it has the form $F := F_1 \wedge F_2 \cdots \wedge F_n$, where each $F_i$ (clause) is a disjunction of literals.

**Example.**
$(\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF.
$\neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF (not in NNF).

**Disjunctive normal form (DNF)** A formula $F$ is in disjunctive normal form iff: Disjunctive normal form

- it is in negation normal form;
- it has the form $F := F_1 \vee F_2 \cdots \vee F_n$, where each $F_i$ is a conjunction of literals.

## 1.3 Reasoning

**Reasoning method** Systems to work with symbols. Reasoning method

Given a set of formulas $\Gamma$, a formula $F$ and a reasoning method $E$, we denote with $\Gamma \vdash^E F$ the fact that $F$ can be deduced from $\Gamma$ using the reasoning method $E$.

**Sound** A reasoning method $E$ is sound iff: Soundness

$$(\Gamma \vdash^E F) \rightarrow (\Gamma \models F)$$

**Complete** A reasoning method $E$ is complete iff: Completeness

$$(\Gamma \models F) \rightarrow (\Gamma \vdash^E F)$$

**Deduction theorem** Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula $G$: Deduction theorem

$$(F_1 \wedge \dots \wedge F_n) \models G \iff \models (F_1 \wedge \dots \wedge F_n) \rightarrow G$$

*Proof.*

$\rightarrow$ ) By hypothesis $(F_1 \wedge \dots \wedge F_n) \models G$.

So, for each interpretation $\mathcal{I}$ in which $(F_1 \wedge \dots \wedge F_n)$ is true, $G$ is also true. Therefore, $\mathcal{I} \models (F_1 \wedge \dots \wedge F_n) \rightarrow G$.

Moreover, for each interpretation $\mathcal{I}'$ in which $(F_1 \wedge \dots \wedge F_n)$ is false, $(F_1 \wedge \dots \wedge F_n) \rightarrow G$ is true. Therefore, $\mathcal{I}' \models (F_1 \wedge \dots \wedge F_n) \rightarrow G$.

In conclusion, $\models (F_1 \wedge \dots \wedge F_n) \rightarrow G$.

$\leftarrow$ ) By hypothesis $\models (F_1 \wedge \dots \wedge F_n) \rightarrow G$. Therefore, for each interpretation where $(F_1 \wedge \dots \wedge F_n)$ is true, $G$ is also true.

In conclusion, $(F_1 \wedge \dots \wedge F_n) \models G$.

$\square$

**Refutation theorem** Given a set of formulas $\{F_1, \dots, F_n\}$ and a formula $G$: Refutation theorem

$$(F_1 \wedge \dots \wedge F_n) \models G \iff F_1 \wedge \dots \wedge F_n \wedge \neg G \text{ is inconsistent}$$

Note: this theorem is not accepted in intuitionistic logic.

*Proof.* By definition, $(F_1 \wedge \dots \wedge F_n) \models G$ iff for every interpretation where $(F_1 \wedge \dots \wedge F_n)$ is true, $G$ is also true. This requires that there are no interpretations where $(F_1 \wedge \dots \wedge F_n)$ is true and $G$ false. In other words, it requires that $(F_1 \wedge \dots \wedge F_n \wedge \neg G)$ is inconsistent. $\square$

### 1.3.1 Natural deduction

**Proof theory** Set of rules that allows to derive conclusions from premises by exploiting syntactic manipulations. Proof theory

**Natural deduction** Set of rules to introduce or eliminate connectives. We consider a subset $\{\wedge, \rightarrow, \bot\}$ of functionally complete connectives. Natural deduction for propositional logic

Natural deduction can be represented using a tree like structure:

$$\dfrac{\begin{array}{c}[\text{hypothesis}]\\ \vdots \\ \text{premise}\end{array}}{\text{conclusion}}\ \text{rule name}$$

The conclusion is true when the hypothesis are able to prove the premise. Another tree can be built on top of premises to prove them.

**Introduction** Usually used to prove the conclusion by splitting it. Introduction rules

$$\dfrac{\psi \qquad \varphi}{\varphi \wedge \psi}\ \wedge\text{I} \qquad\qquad \dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}\ \rightarrow\text{I}$$

**Elimination** Usually used to exploit hypothesis and derive a conclusion. Elimination rules

$$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{E} \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{E} \qquad \dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\ \rightarrow\text{E}$$

**Ex falso sequitur quodlibet** From contradiction, anything follows. This can be used when we have two contradicting hypothesis. Ex falso sequitur quodlibet

$$\dfrac{\bot}{\varphi}\ \bot$$

**Reductio ad absurdum** Assume the opposite and prove a contradiction (not accepted in intuitionistic logic). Reductio ad absurdum

$$\dfrac{\begin{array}{c}[\neg\varphi]\\ \vdots \\ \bot\end{array}}{\varphi}\ \text{RAA}$$

# 2 First order logic

## 2.1 Syntax

The symbols of propositional logic are: Syntax

**Constants** Known elements of the domain. Do not represent truth values.

**Variables** Unknown elements of the domain. Do not represent truth values.

**Function symbols** Function $f^{(n)}$ applied on $n$ constants to obtain another constant.

**Predicate symbols** Function $P^{(n)}$ applied on $n$ constants to obtain a truth value.

**Connectives** $\forall$ $\exists$ $\wedge$ $\vee$ $\rightarrow$ $\neg$ $\leftrightarrow$ $\top$ $\bot$ ( )

Using the basic syntax, the following constructs can be defined:

**Term** Denotes elements of the domain.

$$t := \texttt{constant} \mid \texttt{variable} \mid f^{(n)}(t_1, \dots, t_n)$$

**Proposition** Denotes truth values.

$$P := \top \mid \bot \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid P \leftrightarrow P \mid \neg P \mid \forall x.P \mid \exists x.P \mid (P) \mid P^{(n)}(t_1, \dots, t_n)$$

**Well-formed formula** The definition of well-formed formula in first order logic extends the one of propositional logic by adding the following conditions: Well-formed formula

- If S is well-formed, $\exists X.S$ is well-formed. Where $X$ is a variable.
- If S is well-formed, $\forall X.S$ is well-formed. Where $X$ is a variable.

**Free variables** The universal and existential quantifiers bind their variable within the scope of the formula. Let $F_v(F)$ be the set of free variables in a formula $F$, $F_v$ is defined as follows: Free variables

- $F_v(p(t)) = \bigcup \texttt{vars}(t)$
- $F_v(\top) = F_v(\bot) = \varnothing$
- $F_v(\neg F) = F_v(F)$
- $F_v(F_1 \wedge F_2) = F_v(F_1 \vee F_2) = F_v(F_1 \rightarrow F_2) = F_v(F_1) \cup F_v(F_2)$
- $F_v(\forall X.F) = F_v(\exists X.F) = F_v(F) \smallsetminus \{X\}$

**Closed formula/Sentence** Proposition without free variables. Sentence

**Theory** Set of sentences. Theory

**Ground term/Formula** Proposition without variables. Formula

## 2.2 Semantics

**Interpretation** An interpretation in first order logic $\mathcal{I}$ is a pair $(D, I)$: Interpretation

- $D$ is the domain of the terms.
- $I$ is the interpretation function such that:
  - $I(f) : D^n \to D$ for every n-ary function symbol.
  - $I(p) \subseteq D^n$ for every n-ary predicate symbol.

**Variable evaluation** Given an interpretation $\mathcal{I} = (D, I)$ and a set of variables $\mathcal{V}$, a variable is evaluated through $\eta : \mathcal{V} \to D$. Variable evaluation

**Model** Given an interpretation $\mathcal{I}$ and a formula $F$, $\mathcal{I}$ models $F$ ($\mathcal{I} \models F$) when $\mathcal{I}, \eta \models F$ for every variable evaluation $\eta$. Model

A sentence $S$ is:

**Valid** $S$ is satisfied by every interpretation ($\forall \mathcal{I} : \mathcal{I} \models S$).

**Satisfiable** $S$ is satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \models S$).

**Falsifiable** $S$ is not satisfied by some interpretations ($\exists \mathcal{I} : \mathcal{I} \not\models S$).

**Unsatisfiable** $S$ is not satisfied by any interpretation ($\forall \mathcal{I} : \mathcal{I} \not\models S$).

**Logical consequence** A sentence $T_1$ is a logical consequence of $T_2$ ($T_2 \models T_1$) if every model of $T_2$ is also model of $T_1$: Logical consequence

$$\mathcal{I} \models T_2 \to \mathcal{I} \models T_1$$

**Theorem 2.2.1.** It is undecidable to determine if a first order logic formula is a tautology.

**Equivalence** A sentence $T_1$ is equivalent to $T_2$ if $T_1 \models T_2$ and $T_2 \models T_1$. Equivalence

**Theorem 2.2.2.** The following statements are equivalent:

1. $F_1, \dots, F_n \models G$.
2. $(\bigwedge_{i=1}^{n} F_i) \to G$ is valid.
3. $(\bigwedge_{i=1}^{n} F_i) \land \neg G$ is unsatisfiable.

## 2.3 Substitution

**Substitution** A substitution $\sigma : \mathcal{V} \to \mathcal{T}$ is a mapping from variables to terms. It is written as $\{X_1 \mapsto t_1, \dots, X_n \mapsto t_n\}$. Substitution

The application of a substitution is the following:

- $p(t_1, \dots, t_n)\sigma = p(t_1\sigma, \dots, t_n\sigma)$
- $f(t_1, \dots, t_n)\sigma = fp(t_1\sigma, \dots, t_n\sigma)$
- $\bot\sigma = \bot$ and $\top\sigma = \top$
- $(\neg F)\sigma = (\neg F\sigma)$
- $(F_1 \star F_2)\sigma = (F_1\sigma \star F_2\sigma)$ for $\star \in \{\land, \lor, \to\}$

- $(\forall X.F)\sigma = \forall X'(F\sigma[X \mapsto X'])$ where $X'$ is a fresh variable (i.e. does not appear in $F$).
- $(\exists X.F)\sigma = \exists X'(F\sigma[X \mapsto X'])$ where $X'$ is a fresh variable.

**Unifier** A substitution $\sigma$ is a unifier for $e_1, \ldots, e_n$ if $e_1\sigma = \cdots = e_n\sigma$. Unifier

**Most general unifier** A unifier $\sigma$ is the most general unifier (MGU) for $\bar{e} = e_1, \ldots, e_n$ if every unifier $\tau$ for $\bar{e}$ is an instance of $\sigma$ ($\tau = \sigma\rho$ for some substitution $\rho$). In other words, $\sigma$ is the smallest substitution to unify $\bar{e}$. Most general unifier

# 3 Logic programming

## 3.1 Syntax

A logic program has the following components (defined using BNF):

**Atom** $A := p(t_1, \dots, t_n)$ for $n \geq 0$ Atom

**Goal** $G := \top \mid \bot \mid A \mid G_1 \wedge G_2$ Goal

**Horn clause** A clause with at most one positive literal. Horn clause

$$K := A \leftarrow G$$

In other words, $A$ and all the literals in $G$ are positive as $A \leftarrow G = A \vee \neg G$.

**Program** $P := K_1 \dots K_m$ for $m \geq 0$ Program

## 3.2 Semantics

### 3.2.1 State transition system

**State** Pair $\langle G, \theta \rangle$ where $G$ is a goal and $\theta$ is a substitution. State

**Intial state** $\langle G, \varepsilon \rangle$

**Successful final state** $\langle \top, \theta \rangle$

**Failed final state** $\langle \bot, \varepsilon \rangle$

**Derivation** A sequence of states. A derivation can be: Derivation

**Successful** If the final state is successful.

**Failed** If the final state is failed.

**Infinite** If there is an infinite sequence of states.

Given a derivation, a goal $G$ can be:

**Successful** There is a successful derivation starting from $\langle G, \varepsilon \rangle$.

**Finitely failed** All the derivations starting from $\langle G, \varepsilon \rangle$ are failed.

**Computed answer substitution** Given a goal $G$ and a program $P$, if there exists a successful derivation $\langle G, \varepsilon \rangle \mapsto * \langle \top, \theta \rangle$, then the substitution $\theta$ is the computed answer substitution of $G$. Computed answer substitution

**Transition** Starting from the state $\langle A \wedge G, \theta \rangle$ of a program $P$, a transition on the atom $A$ can result in: Transition

**Unfold** If there exists a clause $(B \leftarrow H)$ in $P$ and a (most general) unifier $\beta$ for $A\theta$ and $B$, then we have a transition: $\langle A \wedge G, \theta \rangle \mapsto \langle H \wedge G, \theta\beta \rangle$.

In other words, we want to prove that $A\theta$ holds. To do this, we search for a clause that has as conclusion $A\theta$ and add its premise to the things to prove. If a unification is needed to match $A\theta$, we add it to the substitutions of the state.

**Failure** If there are no clauses $(B \leftarrow H)$ in $P$ with a unifier for $A\theta$ and $B$, then we have a transition: $\langle A \wedge G, \theta \rangle \mapsto \langle \bot, \varepsilon \rangle$.

**Non-determinism** A transition has two types of non-determinism:

**Don't-care** Any atom in $(A \wedge G)$ can be chosen to determine the next state. This affects the length of the derivation (infinite in the worst case). Don't-care

**Don't-know** Any clause $(B \rightarrow H)$ in $P$ with an unifier for $A\theta$ and $B$ can be chosen. This determines the output of the derivation. Don't-know

**Selective linear definite resolution** Approach to avoid non-determinism when constructing a derivation. SLD resolution

**Selection rule** Method to select the atom in the goal to unfold (eliminates don't-care non-determinism). Selection rule

**SLD tree** Search tree constructed using all the possible clauses according to a selection rule and visited following a search strategy (eliminates don't know non-determinism). SLD tree

**Theorem 3.2.1** (Soundness)**.** Given a program $P$, a goal $G$ and a substitution $\theta$, if $\theta$ is a computed answer substitution, then $P \models G\theta$.

**Theorem 3.2.2** (Completeness)**.** Given a program $P$, a goal $G$ and a substitution $\theta$, if $P \models G\theta$, then it exists a computed answer substitution $\sigma$ such that $G\theta = G\sigma\beta$.

**Theorem 3.2.3.** If a computed answer substitution can be obtained using a selection rule $r$, it can be obtained also using a different selection rule $r'$.

**Prolog SLD** Prolog SLD

**Selection rule** Select the leftmost atom.

**Tree search strategy** Search following the order of definition of the clauses.

This results in a left-to-right, depth-first search of the SLD tree. Note that this may end up in a loop.

# 4 Prolog

It may be useful to first have a look at the "Logic programming" section of `Languages and Algorithms for AI (module 2)`.

## 4.1 Syntax

**Term** Following the first-order logic definition, a term can be a: Term

- Constant (`lowerCase`).
- Variable (`UpperCase`).
- Function symbol (`f(t1, ..., tn)` with `t1`, ..., `tn` terms).

**Atomic formula** An atomic formula has form: Atomic formula

```
p(t1, ..., tn)
```

where `p` is a predicate symbol and `t1`, ..., `tn` are terms.

Note: there are no syntactic distinctions between constants, functions and predicates.

**Clause** A Prolog program is a set of horn clauses: Horn clause

**Fact** `A.`

**Rule** `A :- B1, ..., Bn.` (`A` is the head and `B1, ..., Bn` the body)

**Goal** `:- B1, ..., Bn.`

where:

- `A`, `B1`, ..., `Bn` are atomic formulas.
- `,` represents the conjunction ($\wedge$).
- `:-` represents the logical implication ($\Leftarrow$).

**Quantification** Quantification

**Facts** Variables appearing in a fact are quantified universally.

$$\texttt{A(X).} \equiv \forall \texttt{X} : \texttt{A(X)}$$

**Rules** Variables appearing the the body only are quantified existentially. Variables appearing in both the head and the body are quantified universally.

$$\texttt{A(X) :- B(X, Y).} \equiv \forall \texttt{X}, \exists \texttt{Y} : \texttt{A(X)} \Leftarrow \texttt{B(X, Y)}$$

**Goals** Variables are quantified existentially.

$$\texttt{:- B(Y).} \equiv \exists \texttt{Y} : \texttt{B(Y)}$$

## 4.2 Semantics

**Execution of a program** A computation in Prolog attempts to prove the goal. Given a program $P$ and a goal `:- p(t1, ..., tn)`, the objective is to find a substitution $\sigma$ such that:

$$P \models [\,\texttt{p(t1, ..., tn)}\,]\sigma$$

In practice, it uses two stacks:

**Execution stack** Contains the predicates the interpreter is trying to prove.

**Backtracking stack** Contains the choice points (clauses) the interpreter can try.

**SLD resolution** Prolog uses SLD resolution with the following choices: SLD

**Left-most** Always proves the left-most literal first.

**Depth-first** Applies the predicates following the order of definition.

Note that the depth-first approach can be efficiently implemented (tail recursion) but the termination of a Prolog program on a provable goal is not guaranteed as it may loop depending on the ordering of the clauses.

**Disjunction operator** The operator `;` can be seen as a disjunction and makes the Prolog interpreter explore the remaining SLD tree looking for alternative solutions.

## 4.3 Arithmetic operators

In Prolog: Arithmetic operators

- Integers and floating points are built-in atoms.
- Math operators are built-in function symbols.

Therefore, mathematical expressions are terms.

`is` **predicate** The predicate `is` is used to evaluate and unify expressions:

`T is Expr`

where `T` is a numerical atom or a variable and `Expr` is an expression without free variables. After evaluation, the result of `Expr` is unified with `T`.

**Example.**

```
?- X is 2+3.
    yes X=5
```

Note: a term representing an expression is evaluated only with the predicate `is` (otherwise it remains as is).

**Relational operators** Relational operators (`>`, `<`, `>=`, `=<`, `==`, `=/=`) are built-in.

## 4.4 Lists

A list is defined recursively as: Lists

**Empty list** `[]`

**List constructor** `.(T, L)` where `T` is a term and `L` is a list.

Note that a list always ends with an empty list.
As the formal definition is impractical, some syntactic sugar has been defined:

**List definition** `[t1, ..., tn]` can be used to define a list.

**Head and tail** `[H | T]` where `H` is the head (term) and `T` the tail (list) can be useful for recursive calls.

## 4.5 Cut

The cut operator (`!`) allows to control the exploration of the SLD tree. Cut
A cut in a clause:

```
p :- q1, ..., qi, !, qj, ..., qn.
```

makes the interpreter consider only the first choice points for `q1, ..., qi`, dropping all the other possibilities. Therefore, if `qj, ..., qn` fails, there won't be backtracking and `p` fails.

**Example.**

```
p(X) :- q(X), r(X).
q(1).
q(2).
r(2).

?- p(X).
    yes X=2
```

```
p(X) :- q(X), !, r(X).
q(1).
q(2).
r(2).

?- p(X).
    no
```

In the second case, the cut drops the choice point `q(2)` and only considers `q(1)`.

**Mutual exclusion** A cut can be useful to achieve mutual exclusion. In other words, to represent a conditional branching:

```
if a(X) then b else c
```

a cut can be used as follows:

```
p(X) :- a(X), !, b.
p(X) :- c.
```

If `a(X)` succeeds, other choice points for `p` will be dropped and only `b` will be evaluated. If `a(X)` fails, the second clause will be considered, therefore evaluating `c`.

## 4.6 Negation

**Closed-world assumption** Only what is stated in a program $P$ is true, everything else is false: Closed-world assumption

$$\texttt{CWA}(P) = P \cup \{\neg A \mid A \text{ is a ground atomic formula and } P \not\models A\}$$

**Non-monotonic inference rule** Adding new axioms to the program may change the set of valid theorems.

As first-order logic in undecidable, closed-world assumption cannot be directly applied in practice.

**Negation as failure** A negated atom $\neg A$ is considered true iff $A$ fails in finite time: Negation as failure

$$\texttt{NF}(P) = P \cup \{\neg A \mid A \in \texttt{FF}(P)\}$$

where $\texttt{FF}(P) = \{B \mid P \not\models B \text{ in finite time}\}$ is the set of atoms for which the proof fails in finite time. Note that not all atoms $B$ such that $P \not\models B$ are in $\texttt{FF}(P)$.

**SLDNF** SLD resolution with NF to solve negative atoms. SLDNF

Given a goal of literals `:- `$\texttt{L}_1$`, ..., `$\texttt{L}_m$, SLDNF does the following:

1. Select a positive or ground negative literal $\texttt{L}_i$:
   - If $\texttt{L}_i$ is positive, apply the normal SLD resolution.
   - If $\texttt{L}_i = \neg A$, prove that $A$ fails in finite time. If it succeeds, $\texttt{L}_i$ fails.
2. Solve the goal `:- `$\texttt{L}_1$`, ..., `$\texttt{L}_{i-1}$`, `$\texttt{L}_{i+1}$`, ...`$\texttt{L}_m$.

**Theorem 4.6.1.** If only positive or ground negative literal are selected during resolution, SLDNF is correct and complete.

**Prolog SLDNF** Prolog uses an incorrect implementation of SLDNF where the selection rule always chooses the left-most literal. This potentially causes incorrect deductions.

*Proof.* When proving `:- \+capital(X).`, the intended meaning is:

$$\exists \texttt{X} : \neg\texttt{capital(X)}$$

In SLDNF, to prove `:- \+capital(X).`, the algorithm proves `:- capital(X).`, which results in:

$$\exists \texttt{X} : \texttt{capital(X)}$$

and then negates the result, which corresponds to:

$$\neg(\exists \texttt{X} : \texttt{capital(X)}) \iff \forall \texttt{X} : (\neg\texttt{capital(X)})$$

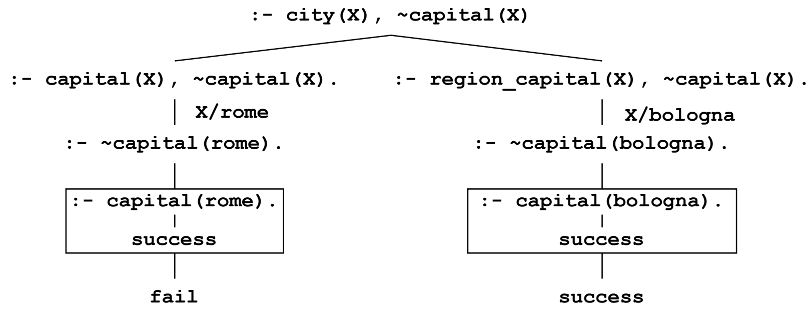
□

**Example** (Correct SLDNF resolution)**.** Given the program:

```
capital(rome).
region_capital(bologna).
city(X) :- capital(X).
city(X) :- region_capital(X).

?- city(X), \+capital(X).
```

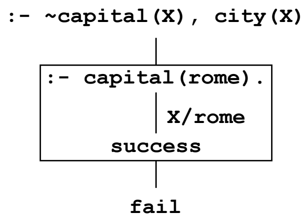
its resolution succeeds with `X=bologna` as `\+capital(X)` is ground by the unification of `city(X)`.




**Example** (Incorrect SLDNF resolution)**.** Given the program:

```
capital(rome).
region_capital(bologna).
city(X) :- capital(X).
city(X) :- region_capital(X).

?- \+capital(X), city(X).
```




its resolution fails as `\+capital(X)` is a free variable and the proof of `capital(X)` is ground with `X=rome` and succeeds, therefore failing `\+capital(X)`. Note that `bologna` is not tried as it does not appear in the axioms of `capital`.

## 4.7 Meta predicates

`call/1` Given a term `T`, `call(T)` considers `T` as a predicate and evaluates it. At the time of evaluation, `T` must be a non-numeric term. `call/1`

**Example.**

```
p(X) :- call(X).
q(a).

?- p(q(Y)).
    yes Y=a
```

`fail/0` The evaluation of `fail` always fails, forcing the interpreter to backtrack. `fail/0`

**Example** (Implementation of negation as failure)**.**

```
not(P) :- call(P), !, fail.
not(P).
```

Note that the cut followed by `fail` (`!, fail`) is useful to force a global failure.

**`bagof/3` and `setof/3`**

`bagof/3` The predicate `bagof(X, P, L)` unifies `L` with a list of the instances of `X` that satisfy `P`. Fails if none exists. `bagof/3`

`sefof/3` The predicate `setof(X, P, S)` unifies `S` with a set of the instances of `X` that satisfy `P`. Fails if none exists. `sefof/3`

In practice, for computational reasons, a list (with repetitions) might be computed.

**Example.**

```
p(1).
p(2).
p(1).

?- setof(X, p(X), S).
    yes S=[1, 2] X=X

?- bagof(X, p(X), S).
    yes S=[1, 2, 1] X=X
```

**Quantification** When solving a goal, the interpreter unifies free variables with a value. This may cause unwanted behaviors when using `bagof` or `setof`. The `X^` tells the interpreter to not (permanently) bind the variable `X`.

**Example.**

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).

?- setof(X, father(X, Y), S).
    yes X=X Y=giuseppe S=[giovanni];
        X=X Y=mario    S=[giovanni];
        X=X Y=paola    S=[mario]
```

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).
?- setof(X, Y^father(X, Y), S).
    yes S=[giovanni, mario] X=X Y=Y
```

`findall/3` The predicate `findall(X, P, S)` unifies `S` with a list of the instances of `X` that satisfy `P`. If none exists, `S` is unified with an empty list. Variables in `P` that do not appear in `X` are not bound (same as the `Y^` operator). findall/3

**Example.**

```
father(giovanni, mario).
father(giovanni, giuseppe).
father(mario, paola).

?- findall(X, father(X, Y), S).
    yes S=[giovanni, mario] X=X Y=Y
```

`var/1` The predicate `var(T)` is true if `T` is a variable. var/1

`nonvar/1` The predicate `nonvar(T)` is true if `T` is not a free variable. nonvar/1

`number/1` The predicate `number(T)` is true if `T` is a number. number/1

`ground/1` The predicate `ground(T)` is true if `T` does not have free variables. ground/1

`=../2` The operator `T =.. L` unifies `L` with a list where its head is the head of `T` and the tail contains the remaining arguments of `T` (i.e. puts all the components of a predicate into a list). Only one between `T` and `L` may be a variable. =../2

**Example.**

```
?- foo(hello, X) =.. List.
    List = [foo, hello, X]
```

```
?- Term =.. [baz, foo(1)].
    Term = baz(foo(1))
```

clause/2

**clause/2** The predicate `clause(Head, Body)` is true if it can unify `Head` and `Body` with an existing clause. `Head` must be initialized to a non-numeric term. `Body` can be a variable or a term.

**Example.**

```
p(1).
q(X, a) :- p(X), r(a).
q(2, Y) :- d(Y).

?- clause(p(1), B).
     yes B=true

?- clause(p(X), true).
     yes X=1

?- clause(q(X, Y), B).
     yes X=_1 Y=a B=p(_1), r(a);
         X=2 Y=_2 B=d(_2)
```

assert/1

**assert/1** The predicate `assert(T)` adds `T` in an unspecified position of the clauses database of Prolog. In other words, it allows to dynamically add clauses.

asserta/1

**asserta/1** As `assert(T)`, with insertion at the beginning of the database.

assertz/1

**assertz/1** As `assert(T)`, with insertion at the end of the database.

Note that `:- assert((p(X)))` quantifies `X` existentially as it is a query. If it is not ground and added to the database as is, is becomes a clause and therefore quantified universally: $\forall$`X` : `p(X)`.

**Example** (Lemma generation)**.**

```
fib(0, 0) :- !.
fib(1, 1) :- !.
fib(N, F) :- N1 is N-1, fib(N1, F1),
             N2 is N-2, fib(N2, F2),
             F is F1+F2,
             generate_lemma(fib(N, F)).

generate_lemma(T) :- clause(T, true), !.
generate_lemma(T) :- assert(T).
```

`generate_lemma/1` allows to add to the clauses database all the intermediate steps to compute the Fibonacci sequence (similar concept to dynamic programming).

retract/1

**retract/1** The predicate `retract(T)` removes from the database the first clause that unifies with `T`.

abolish/2

**abolish/2** The predicate `abolish(T, n)` removes from the database all the occurrences of `T` with arity `n`.

## 4.8 Meta-interpreters

Meta-interpreter

**Meta-interpreter** Interpreter for a language $L_1$ written in another language $L_2$.

Vanilla meta-interpreter

**Prolog vanilla meta-interpreter** The Prolog vanilla meta-interpreter is defined as follows:

```
solve(true) :- !.
solve( (A, B) ) :- !, solve(A), solve(B).
solve(A) :- clause(A, B), solve(B).
```

In other words, the clauses state the following:

1. A tautology is a success.
2. To prove a conjunction, we have to prove both atoms.
3. To prove an atom `A`, we look for a clause `A :- B` that has `A` as conclusion and prove its premise `B`.

# 5 Constraint programming

**Class of problems**

**Constraint satisfaction problem (CSP)** Defined by:

Constraint satisfaction problem

- A finite set of variables $X_1, \dots, X_n$.
- A domain for each variable $D(X_1), \dots, D(X_n)$.
- A set of constraints $\{C_1, \dots, C_m\}$

A solution is an assignment to all the variables while satisfying the constraints.

**Constraint optimization problem (COP)** Extension of a constraint satisfaction problem with an objective function with domain $D$:

Constraint optimization problem

$$f : D(X_1) \times \cdots \times D(X_n) \to D$$

A solution is a CSP solution that optimizes $f$.

**Class of languages**

**Constraint logic programming (CLP)** Add constraints and solvers to logic programming. Generally more efficient than plain logic programming.

Constraint logic programming

**Imperative languages** Add constraints and solvers to imperative languages.

Imperative languages

## 5.1 Constraint logic programming (CLP)

### 5.1.1 Syntax

**Atom** $A := p(t_1, \dots, t_n)$, for $n \geq 0$. $p$ is a predicate.

Atom

**Constraint** $C := c(t_1, \dots, t_n) \mid C_1 \wedge C_2$, for $n \geq 0$. $c$ is an atomic constraint.

Constraint

**Goal** $G := \top \mid \bot \mid A \mid C \mid G_1 \wedge G_2$

Goal

**Constraint logic clause** $K := A \leftarrow G$

Constraint logic clause

**Constraint logic program** $P := K_1 \dots K_m$, for $m \geq 0$

Constraint logic program

### 5.1.2 Semantics

**Transition system**

Transition system

**State** Pair $\langle G, C \rangle$ where $G$ is a goal and $C$ is a constraint.

**Initial state** $\langle G, \top \rangle$

**Successful final state** $\langle \top, C \rangle$ with $C \neq \bot$

**Failed final state** $\langle G, \bot \rangle$

**Transition** Starting from the state $\langle A \wedge G, C \rangle$ of a program $P$, a transition on the atom $A$ can result in:

- **Unfold** If there exists a clause $(B \leftarrow H)$ in $P$ and an assignment $(B \doteq A)$ such that $((B \doteq A) \wedge C)$ is still valid, then we have a transition $\langle A \wedge G, C\rangle \mapsto \langle H \wedge G, (B \doteq A) \wedge C\rangle$. Unfold

  In other words, we want to develop an atom $A$ and the current constraints are denoted as $C$. We look for a clause whose head equals $A$, applying an assignment if needed. If this is possible, we transit from solving $A$ to solving the body of the clause and add the assignment to the set of active constraints.

- **Failure** If there are no clauses $(B \leftarrow H)$ with a valid assignment $((B \doteq A) \wedge C)$, then we have a transition $\langle A \wedge G, C\rangle \mapsto \langle \bot, \bot\rangle$. Failure

Moreover, starting from the state $\langle C \wedge G, D_1\rangle$ of a program $P$, a transition on the constraint $C$ can result in:

- **Solve** If $(C \wedge D_1) \iff D_2$ holds, then we have a transition $\langle C \wedge G, D_1\rangle \mapsto \langle G, D_2\rangle$. Solve

  In other words, we want to develop a constraint $C$ and the current constraints are denoted as $D_1$. If $(C \wedge D_1)$ is valid, we call it $D_2$ and continue solving the rest of the goal constrained to $D_2$.

**Non-determinism** As in logic programming, there is don't-care and don't-know non-determinism. A SLD search tree is also used.

**Derivation strategies**

**Generate-and-test** Strategy adopted by logic programs. Every possible assignment to the variables are generated and tested. Generate-and-test

The workflow is the following:

1. Determine domain.
2. Make an assignment to each variable.
3. Test the constraints.

**Constrain-and-generate** Strategy adopted by constraint logic programs. Exploit facts to reduce the search space. Constrain-and-generate

The workflow is the following:

1. Determine domain.
2. Restrict the domain following the constraints.
3. Make an assignment to each variable.

## 5.2 MiniZinc

Declarative language for constraint programming.

**Built-in types** `bool`, `int`, `float`, `string` Built-in types

**Logical operators** `\/` · `/\` · `->` · `!` Logical operators

**Parameter** User-inputted value passed to the solver before execution. Parameter

```
<domain>: <name>
```

**Example.** `int: size;`

**Variable** Value computed by the solver. Variable

```
var <domain>: <name>
```

**Example.** `var bool: flag;`

**Set** For defining ranges. Set

```
set of <domain>: <name>
```

**Example.** `set of int: top10 = 1..10;`

**Array** Array of parameters or variables. Array

```
array[<index range>] of <domain>: <name>
```

**Example.** `array[1..5] of var int: vars;`

**Aggregation functions** `sum`, `product`, `min`, `max`. Aggregation functions

**Forall**

```
forall(<iterators> in <domain>)(<conditions>)
forall(<iterators> in <domain> where <conditions>)(<conditions>)
```

**Example.** `forall(i, j in 1..3 where i < j)(arr[i] != arr[j]);`

**Exists**

```
exists(<iterators> in <domain>)(<conditions>)
exists(<iterators> in <domain> where <conditions>)(<conditions>)
```

**Constraints** Constraints

```
constraint <expression>
```

Multiple constraints are seen as conjunctions.

**Example.** `constraint X >= 5 /\ X != 10;`

**Global constraints** `all_different(...)`, `all_equal(...)`

**Solver** Solver

**Satisfiability problem**

```
solve satisfy;
```

**Optimization problem**

```
solve minimize <variable>;
```

`<end of course>`