Combinatorial Decision Making and Optimization (Module 2)

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Contents

1 Satisfiability modulo theory

Satisfiability modulo theory (SMT) Satisfiability of a formula with respect to some background formal theory/theories. Satisfiability modulo theory (SMT)

SMT extends SAT and exploits domain-specific reasoning (possibly with infinite domains).

1.1 First-order logic for SMT

1.1.1 Syntax

Remark. Only quantifier-free formulas (q.f.f.) are considered in SMT.

Functions The set of all the functions is denoted as $\Sigma^F = \bigcup_{k\geq 0} \Sigma_k^F$ where Σ_k^F denotes the set of k-ary functions. Functions

Constants Σ_0^F

Predicates The set of all the predicates is denoted as $\Sigma^P = \bigcup_{k\geq 0} \Sigma_k^P$ where Σ_k^P denotes the set of k-ary predicates. Predicates

Propositional symbols Σ_0^P

Signature The set of the non-logical symbols of FOL is denoted as: Signature

$$\Sigma = \Sigma^F \cup \Sigma^P$$

Terms The set of terms over Σ is denoted as $\mathbb{T}^\Sigma$: Terms

$$\begin{aligned}\mathbb{T}^\Sigma = &\Sigma_0^F \cup \\ &\{f(t_1,\ldots,t_k) \mid f \in \Sigma_k^F \wedge t_1,\ldots,t_k \in \mathbb{T}^\Sigma\} \cup \\ &\{\texttt{ite}(\varphi,t_1,t_2) \mid \varphi \in \mathbb{F}^\Sigma \wedge t_1,t_2 \in \mathbb{T}^\Sigma\}\end{aligned}$$

Remark. `ite` is an auxiliary function to capture the if-then-else construct.

Formulas The set of formulas over Σ is denoted as $\mathbb{F}^\Sigma$: Formulas

$$\begin{aligned}\mathbb{F}^\Sigma = &\{\bot,\top\} \cup \Sigma_0^P \cup \\ &\{t_1 = t_2 \mid t_1,t_2 \in \mathbb{T}^\Sigma\} \cup \\ &\{p(t_1,\ldots,t_k) \mid p \in \Sigma_k^P \wedge t_1,\ldots,t_k \in \mathbb{T}^\Sigma\} \cup \\ &\{\neg\varphi \mid \varphi \in \mathbb{F}^\Sigma\} \cup \\ &\{(\varphi_1 \Rightarrow \varphi_2),(\varphi_1 \iff \varphi_2),(\varphi_1 \wedge \varphi_2),(\varphi_1 \vee \varphi_2) \mid \varphi_1,\varphi_2 \in \mathbb{F}^\Sigma\}\end{aligned}$$

1.1.2 Semantics

Σ-model Pair $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ defined on a given signature Σ where: Σ-model

- M is the universe of $\mathcal{M}$.
- $(\cdot)^{\mathcal{M}}$ is a mapping such that:
  - $\forall f \in \Sigma_k^F : f^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to M\}$.
  - $\forall p \in \Sigma_k^P : p^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to \{\texttt{true}, \texttt{false}\}\}$.

Interpretation Extension of the mapping function $(\cdot)^{\mathcal{M}}$ to terms and formulas: Interpretation

- $\top^{\mathcal{M}} = \texttt{true}$ and $\bot^{\mathcal{M}} = \texttt{false}$.
- $(f(t_1, \ldots, t_k))^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_k^{\mathcal{M}})$ and $(p(t_1, \ldots, t_k))^{\mathcal{M}} = p^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_k^{\mathcal{M}})$.
- $\texttt{ite}(\varphi, t_1, t_2)^{\mathcal{M}} = \begin{cases} t_1^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = \texttt{true} \\ t_2^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = \texttt{false} \end{cases}$.

1.1.3 Σ-theory

Satisfiability A model $\mathcal{M}$ satisfies a formula $\varphi \in \mathbb{F}^{\Sigma}$ if $\varphi^{\mathcal{M}} = \texttt{true}$. Satisfiability

Σ-theory Possibly infinite set $\mathcal{T}$ of Σ-models. Σ-theory

$\mathcal{T}$-satisfiability A formula $\varphi \in \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-satisfiable if there exists a model $\mathcal{M} \in \mathcal{T}$ that satisfies it. $\mathcal{T}$-satisfiability

$\mathcal{T}$-consistency A set of formulas $\{\varphi_1, \ldots, \varphi_k\} \subseteq \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-consistent iff $\varphi_1 \wedge \cdots \wedge \varphi_k$ is $\mathcal{T}$-satisfiable. $\mathcal{T}$-consistency

$\mathcal{T}$-entailment A set of formulas $\Gamma \subseteq \mathbb{F}^{\Sigma}$ $\mathcal{T}$-entails a formula $\varphi \in \mathbb{F}^{\Sigma}$ ($\Gamma \models_{\mathcal{T}} \varphi$) iff in every model $\mathcal{M} \in \mathcal{T}$ that satisfies Γ, φ is also satisfied. $\mathcal{T}$-entailment

Remark. Γ is $\mathcal{T}$-consistent iff $\Gamma \not\models_{\mathcal{T}} \bot$.

$\mathcal{T}$-validity A formula $\varphi \in \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-valid iff $\varnothing \models_{\mathcal{T}} \varphi$. $\mathcal{T}$-validity

Remark. φ is $\mathcal{T}$-consistent iff $\neg\varphi$ is not $\mathcal{T}$-valid.

Theory lemma $\mathcal{T}$-valid clause $c = l_1 \vee \cdots \vee l_k$. Theory lemma

Σ-expansion Given a Σ-model $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ and $\Sigma' \supseteq \Sigma$, an expansion $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ over Σ' is any Σ'-model such that: Σ-expansion

- $M' = M$.
- $\forall s \in \Sigma : s^{\mathcal{M}'} = s^{\mathcal{M}}$

Remark. Given a Σ-theory $\mathcal{T}$, we implicitly consider it to be the theory $\mathcal{T}'$ defined as:

$$\mathcal{T}' = \{\mathcal{M}' \mid \mathcal{M}' \text{ is an expansion of a } \Sigma\text{-model } \mathcal{M} \text{ in } \mathcal{T}\}$$

Ground $\mathcal{T}$-satisfiability Given a Σ-theory $\mathcal{T}$, determine if a ground formula is $\mathcal{T}$-satisfiable over a Σ-expansion $\mathcal{T}'$. Ground $\mathcal{T}$-satisfiability

Axiomatically defined theory Given a minimal set of formulas (axioms) $\Lambda \subseteq \mathbb{F}^{\Sigma}$, its corresponding theory is the set of all the models that respect Λ. Axiomatically defined theory

Example. Let Σ be defined as:

$$\Sigma_0^F = \{a, b, c, d\} \qquad \Sigma_1^F = \{f, g\} \qquad \Sigma_2^P = \{p\}$$

A Σ-model $\mathcal{M} = \langle [0, 2\pi[, (\cdot)^{\mathcal{M}} \rangle$ can be defined as follows:

$$a^{\mathcal{M}} = 0 \qquad b^{\mathcal{M}} = \frac{\pi}{2} \qquad c^{\mathcal{M}} = \pi \qquad d^{\mathcal{M}} = \frac{3\pi}{2}$$
$$f^{\mathcal{M}} = \sin \qquad g^{\mathcal{M}} = \cos \qquad p^{\mathcal{M}}(x, y) \iff x > y$$

To determine if $p(g(x), f(d))$ is $\mathcal{M}$-satisfiable, we have to expand $\mathcal{M}$ as there are free variables (x). Let $\Sigma' = \Sigma \cup \{x\}$. The expansion $\mathcal{M}'$ such that $x^{\mathcal{M}'} = \frac{\pi}{2}$ makes the formula satisfiable.

1.1.4 Theories of interest

Equality with Uninterpreted Functions theory (EUF) Theory $\mathcal{T}_{\text{EUF}}$ containing all the possible Σ-models. Equality with Uninterpreted Functions theory (EUF)

Remark. Also called empty theory as its axiom set is $\varnothing$ (i.e. allows any model).

Remark. Useful to deal with black-box functions (i.e. prove satisfiability without a specific theory).

Example. The following formula can be proved to be unsatisfiable by only using syntactic manipulations of basic FOL concepts:

$$\big(a * (f(b) + f(c)) = d\big) \wedge \big(b * (f(a) + f(c)) \neq d\big) \wedge \underline{\big(a = b\big)}$$
$$\underline{\big(a * (f(a) + f(c))\big)} = d\big) \wedge \big(\underline{a * (f(a) + f(c))} \neq d\big)$$
$$\big(g(a, c) = d\big) \wedge \big(g(a, c) \neq d\big)$$

Arithmetic theories Theories with $\Sigma = (0, 1, +, -, \leq)$. Arithmetic theories

Presburger arithmetic Theory $\mathcal{T}_{\mathbb{Z}}$ that interprets Σ-symbols over integers.

- Ground $\mathcal{T}_{\mathbb{Z}}$-satisfiability is **NP**-complete.
- Extended with multiplication, $\mathcal{T}_{\mathbb{Z}}$-satisfiability becomes undecidable.

Real arithmetic Theory $\mathcal{T}_{\mathbb{R}}$ that interprets Σ-symbols over reals.

- Ground $\mathcal{T}_{\mathbb{R}}$-satisfiability is in **P**.
- Extended with multiplication, $\mathcal{T}_{\mathbb{R}}$-satisfiability becomes doubly-exponential.

Remark. In floating points, commutativity still holds, but associativity and distributivity are not guaranteed.

Array theory Let $\Sigma_{\mathcal{A}}$ be the signature containing two functions: Array theory

$\texttt{read}(a, i)$ Reads the value of a at index i.

$\texttt{write}(a, i, v)$ Returns an array a' where the value v is at the index i of a.

The theory $\mathcal{T}_{\mathcal{A}}$ is the set of all models respecting the following axioms:

- $\forall a \, \forall i \, \forall v : \texttt{read}(\texttt{write}(a, i, v), i) = v$.
- $\forall a \, \forall i \, \forall j \, \forall v : (i \neq j) \Rightarrow \Big(\texttt{read}\big(\texttt{write}(a, i, v), j\big) = \texttt{read}(a, j)\Big)$.
- $\forall a \, \forall a' : \big(\forall i : \texttt{read}(a, i) = \texttt{read}(a', i)\big) \Rightarrow (a = a')$.

Remark. The full $\mathcal{T}_{\mathcal{A}}$ theory is undecidable but there are decidable fragments.

Bit-vectors theory Theory $\mathcal{T}_{\mathcal{BV}}$ with vectors of bits of fixed length as constants and operations such as: Bit-vectors theory

- String-like operations (e.g. slicing, concatenation, ...).
- Logical operations (e.g. bit-wise operators).
- Arithmetic operations (e.g. $+$, $-$, ...).

String theory Theory to handle strings of unbounded length. String theory

Theory of word equations Given an alphabet $\mathcal{S}$, a word equation has form $L = R$ where L and R are concatenations of string constants over $\mathcal{S}^*$.

Remark. The general theory of word equations is undecidable.

Remark. The quantifier-free theory of word equations is decidable.

Remark. In practice, many theories are often combined.

1.2 Encoding to SAT

1.2.1 Eager approaches

All the information on the formal theory is used from the beginning to encode an SMT formula φ into an equisatisfiable SAT formula φ' (i.e. SMT is compiled into SAT).

Equisatisfiability Given a Σ-theory $\mathcal{T}$, two formulas φ and φ' are equisatisfiable iff: Equisatisfiability

$$\varphi \text{ is } \mathcal{T}\text{-satisfiable} \iff \varphi' \text{ is } \mathcal{T}\text{-satisfiable}$$

Eager approaches have the following advantages:

- Does not require an SMT solver.
- Once encoded, whichever SAT solver can be used.

Eager approaches have the following disadvantages:

- An ad-hoc encoding is needed for all the theories.
- The resulting SAT formula might be huge.

Algorithm Given an EUF formula φ, to determine if it is $\mathcal{T}_{\text{EUF}}$-satisfiable, the following steps are taken:

1. Replace functions and predicates with constant equalities. Given the terms $f(t_1), \ldots, f(t_k)$, possible approaches are:

   Ackermann approach Ackermann approach

   - Each $f(t_i)$ is encoded into a new constant A_i.
   - Add the constraints $(t_i = t_j) \Rightarrow (A_i = A_j)$ for each $i < j$.

   Bryant approach Bryant approach

   - $f(t_1)$ is encoded as A_1.
   - $f(t_2)$ is encoded as $\texttt{ite}(t_2 = t_1, A_1, A_2)$.

- $f(t_3)$ is encoded as $\texttt{ite}\big(t_3 = t_1, A_1, \texttt{ite}(t_3 = t_2, A_2, A_3)\big)$.
- $f(t_i)$ is encoded as:

$$\texttt{ite}\Big(t_i = t_1, A_1, \texttt{ite}\Big(t_i = t_2, A_2, \texttt{ite}\big(\ldots, \texttt{ite}(t_i = t_{i-1}, A_{i-1}, A_i)\big)\Big)\Big)$$

2. Remove equalities to reduce φ into propositional logic. Possible encodings are:

   Small-domain encoding If φ has n distinct variables $\{c_1, \ldots, c_n\}$, a possible model $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ that satisfies it must have $|M| \leq n$.

   Therefore, each $c_i^{\mathcal{M}}$ can be associated to a value in $\{1, \ldots, n\}$. In SAT, this mapping from $c_i^{\mathcal{M}}$ to $\{1, \ldots, n\}$ can be encoded using $O(\log n)$ bits. Finally, an equality $c_i = c_j$ (or $c_i \neq c_j$) can be encoded by adding bitwise constraints.

   Direct encoding Encode each equality $a = b$ with a propositional symbol $P_{a,b}$ and add transitivity constraints of form $(P_{a,b} \wedge P_{b,c}) \Rightarrow P_{a,c}$.

1.2.2 Lazy approaches

Integrate SAT solvers with theory-specific decision procedures.
These approaches are more flexible and modular and avoid an explosion of SAT clauses. On the other hand, the search becomes SAT-driven and not theory-driven.

Remark. Most SMT solvers follow a lazy approach.

Algorithm Let $\mathcal{T}$ be a theory. Given a conjunction of $\mathcal{T}$-literals $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_n$, to determine its $\mathcal{T}$-satisfiability, a generic lazy solver does the following:

1. Each SMT literal φ_i is encoded (abstracted) into a SAT literal l_i to form the abstraction $\Phi = \{l_1, \ldots, l_n\}$ of φ.
2. The $\mathcal{T}$-solver sends Φ to the SAT-solver.
   - If the SAT-solver determines that Φ is unsatisfiable, then φ is $\mathcal{T}$-unsatisfiable.
   - Otherwise, the SAT-solver returns a model $\mathcal{M} = \{a_1, \ldots, a_n\}$ (an assignment of the literals, possibly partial).
3. The $\mathcal{T}$-solver determines if $\mathcal{M}$ is $\mathcal{T}$-consistent.
   - If it is, then φ is $\mathcal{T}$-satisfiable.
   - Otherwise, update $\Phi = \Phi \cup \neg\mathcal{M}$ and go to Point 2.

Example. Consider the EUF formula φ:

$$\big(g(a) = c\big) \wedge \big((f(g(a)) \neq f(c)) \vee (g(a) = d)\big) \wedge \big(c \neq d\big)$$

- φ abstracted into SAT is:

$$\underbrace{\big(g(a) = c\big)}_{l_1} \wedge \big(\neg \underbrace{\big(f(g(a)) = f(c)\big)}_{l_2} \vee \underbrace{\big(g(a) = d\big)}_{l_3}\big) \wedge \neg \underbrace{\big(c = d\big)}_{l_4}$$

$$l_1 \wedge (\neg l_2 \vee l_3) \wedge \neg l_4$$

  Therefore, $\Phi = \{l_1, (\neg l_2 \vee l_3), \neg l_4\}$

- The $\mathcal{T}$-solver sends Φ to the SAT-solver. Let's say that it return $\mathcal{M} = \{l_1, \neg l_2, \neg l_4\}$.

- The $\mathcal{T}$-solver checks if $\mathcal{M}$ is consistent. Let's say it is not. Let $\Phi' = \Phi \cup \neg\mathcal{M} = \{l_1, (\neg l_2 \vee l_3), \neg l_4, (\neg l_1 \vee l_2 \vee l_4)\}$.
- The $\mathcal{T}$-solver sends Φ' to the SAT-solver. Let's say that it return $\mathcal{M}' = \{l_1, l_2, l_3, \neg l_4\}$.
- The $\mathcal{T}$-solver checks if $\mathcal{M}'$ is consistent. Let's say it is not. Let $\Phi'' = \Phi' \cup \neg\mathcal{M}' = \{l_1, (\neg l_2 \vee l_3), \neg l_4, (\neg l_1 \vee l_2 \vee l_4), (\neg l_1 \vee \neg l_2 \vee \neg l_3 \vee l_4)\}$.
- The $\mathcal{T}$-solver sends Φ'' to the SAT-solver and it detects the unsatisfiability. Therefore, φ is $\mathcal{T}$-unsatisfiable.

Optimizations

- Check $\mathcal{T}$-consistency on partial assignments.
- Given a $\mathcal{T}$-inconsistent assignment μ, find a smaller $\mathcal{T}$-inconsistent assignment $\eta \subseteq \mu$ and add $\neg\eta$ to Φ instead of $\neg\mu$.
- When reaching $\mathcal{T}$-inconsistency, backjump to a $\mathcal{T}$-consistent point in the computation.

1.3 CDCL($\mathcal{T}$)

Lazy solver based on CDCL for SAT extended with a $\mathcal{T}$-solver. The $\mathcal{T}$-solver does the following: CDCL($\mathcal{T}$)

- Checks the $\mathcal{T}$-consistency of a conjunction of literals.
- Performs deduction of unassigned literals.
- Explains $\mathcal{T}$-inconsistent assigments.
- Allows to backtrack.

State transition Transition system to describe the reasoning of SAT or SMT solvers. A transition has form: State transition

$$(\mu \| \varphi) \to (\mu' \| \varphi')$$

where:

- φ and φ' are $\mathcal{T}$-formulas.
- μ and μ' are (partial) boolean assignments to atoms of φ and φ', respectively.
- $(\mu \| \varphi)$ and $(\mu' \| \varphi')$ are states.

Transition rule Determine the possible transitions.

Derivation Sequence of transitions.

Initial state $(\varnothing \| \varphi)$.

$\mathcal{T}$-consistency Given a $\mathcal{T}$-formula φ and a full assignment μ of φ, φ is $\mathcal{T}$-consistent ($\mu \models_{\mathcal{T}} \varphi$) if there is a derivation from $(\varnothing \| \varphi)$ to $(\mu \| \varphi)$.

$\mathcal{T}$-propagation Deduce the assignment of an unassigned literal l using some knowledge of the theory. $\mathcal{T}$-propagation

$\mathcal{T}$-consequence If: $\mathcal{T}$-consequence

- $\mu \models_{\mathcal{T}} l$,

- l or $\neg l$ occur in φ,
- l and $\neg l$ do not occur in μ,

then:

$$(\mu \| \varphi) \to (\mu \cup \{l\} \| \varphi)$$

Example. Given the formula φ:

$$\big(g(a) = c\big) \wedge \Big(\big(f(g(a)) \neq f(c)\big) \vee \big(g(a) = d\big)\Big) \wedge \big(c \neq d\big)$$

A possible derivation for some theory $\mathcal{T}$ (i.e. $\mathcal{T}$-propagation are decided arbitrarily) is:

1. $\varnothing \| \varphi$ (initial state).
2. $\varnothing \| \varphi \to \{l_1\} \| \varphi$ (Unit propagation).
3. $\{l_1\} \| \varphi \to \{l_1, l_2\} \| \varphi$ ($\mathcal{T}$-propagation).
4. $\{l_1, l_2\} \| \varphi \to \{l_1, l_2, l_3\} \| \varphi$ (Unit propagation).
5. $\{l_1, l_2, l_3\} \| \varphi \to \{l_1, l_2, l_3, l_4\} \| \varphi$ ($\mathcal{T}$-propagation).
6. $\{l_1, l_2, l_3, l_4\} \| \varphi \to$ `fail` (Failure).

As we are at decision level 0 (as no decision literal has been fixed), we can conclude that φ is unsatisfiable.

Remark. Unit and theory propagation are alternated (see algorithm description).

Algorithm Given a $\mathcal{T}$-formula φ and a (partial) $\mathcal{T}$-assignment μ (i.e. initial decisions), CDCL($\mathcal{T}$) does the following:

Algorithm 1 CDCL($\mathcal{T}$)

```
def cdclT(φ, μ):
    if preprocess(φ, μ) == CONFLICT: return UNSAT
    φ^p, μ^p = SMT_to_SAT(φ), SMT_to_SAT(μ)
    level = 0
    l = None

    while True:
        status = propagate(φ^p, μ^p, l)
        if status == SAT:
            return SAT_to_SMT(μ^p)
        elif status == UNSAT:
            η^p, jump_level = analyzeConflict(φ^p, μ^p)
            if jump_level < 0: return UNSAT
            backjump(jump_level, φ^p ∧ ¬η^p, μ^p)
        elif status == UNKNOWN:
            l = decideNextLiteral(φ^p, μ^p)
            level += 1
```

Where:

`preprocess` Preprocesses φ with μ through operations such as simplifications, $\mathcal{T}$-specific rewritings, ...

`SMT_TO_SAT` Provides the boolean abstraction of an SMT formula.

`SAT_TO_SMT` Reverses the boolean abstraction of an SMT formula.

`propagate` Iteratively apply:

- Unit propagation,
- $\mathcal{T}$-consistency check,
- $\mathcal{T}$-propagation.

Returns `SAT`, `UNSAT` or `UNKNOWN` (when no deductions are possible and there are still free variables).

`analyzeConflict` Performs conflict analysis:

- If the conflict is detected by SAT boolean propagation ($\mu^p \wedge \varphi^p \models_p \bot$), a boolean conflict set η^p is outputted (as in standard CDCL).
- If the conflict is detected by $\mathcal{T}$-propatation ($\mu \wedge \phi \models_{\mathcal{T}} \bot$), a theory conflict η is produced and its boolean abstraction η^p is outputted.

Moreover, the earliest decision level at which a variable of η^p is unassigned is returned.

As in standard CDCL, $\neg\eta^p$ is added to φ^p and the algorithm backjumps to a previous decision level (if possible).

`decideNextLiteral` Decides the assignment of an unassigned variable (as in standard CDCL). Theory information might be exploited.

Implication graph As in the standard CDCL algorithm, an implication graph is used to explain conflicts. Implication graph

Nodes Decisions, derived literals or conflicts.

Edges If v allows to unit/theory propagate w, then there is an edge $v \to w$.

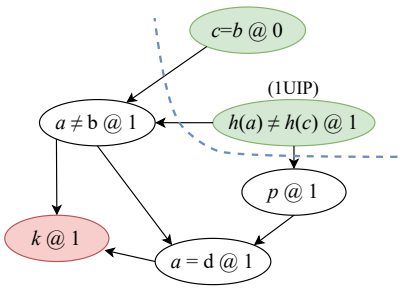
Example. Given the $\mathcal{T}$-formula φ:

$$\big(h(a) = h(c) \vee p\big) \wedge \big(a = b \vee \neg p \vee a = d\big) \wedge \big(a \neq d \vee a = b\big)$$

and an initial decision $(c = b) \in \mu$, CDCL($\mathcal{T}$) does the following:

1. As no propagation is possible, the decision $h(a) \neq h(c)$ is added to μ.
2. Unit propagate p due to the clause $\big(h(a) = h(c) \vee p\big)$ and the decision at the previous step.
3. $\mathcal{T}$-propagate $(a \neq b)$ due to the current assigments: $\{c = b, h(a) \neq h(c)\} \models_{\mathcal{T}} a \neq b$.
4. Unit propagate $(a = d)$ due to the clause $\big(a = b \vee \neg p \vee a = d\big)$ and the current knowledge base (p and $a \neq b$).
5. There is a conflict between $(a \neq d)$ and $(a = d)$.

By building the conflict graph, one can identify the 1UIP as the decision $h(a) \neq h(c)$.



A cut in front of the 1UIP that separates decision nodes and the conflict node (as in standard CDCL) is made to obtain the conflict set $\eta = \{h(a) \neq h(c), c = b\}$. $\big((h(a) = h(c)) \vee (c \neq b)\big)$ is added as a clause and the algorithm backjumps at the decision level 0.

1.4 Theory solvers

Decide satisfiability of theory-specific formulas.

1.4.1 EUF theory

Congruence closure Given a conjunction of EUF literals Φ, its satisfiability can be decided in polynomial time as follows: Congruence closure

1. Add a new variable c and replace each $p(t_1, \ldots, t_k)$ with $f_p(t_1, \ldots, t_k) = c$.
2. Partition input literals into the sets of equalities E and disequalities D.
3. Define E^* as the congruence closure of E. It is the smallest equivalence relation $\equiv_E$ over terms such that:
   - $(t_1 = t_2) \in E \Rightarrow (t_1 \equiv_E t_2)$.
   - For each $f(s_1, \ldots, s_k)$ and $f(t_1, \ldots, t_k)$ occurring in E, if $s_i \equiv_E t_i$ for each $i \in \{1, \ldots, k\}$, then $f(s_1, \ldots, s_k) \equiv_E f(t_1, \ldots, t_k)$.
4. Φ is satisfiable iff $\forall (t_1 \neq t_2) \in D \Rightarrow (t_1 \not\equiv_E t_2)$.

Remark. In practice, congruence closure is usually implemented using a DAG to represent terms and union-find for the E^* class.

1.4.2 Arithmetic theories

Linear real arithmetic LRA theory has signature $\Sigma_{\text{LRA}} = (\mathbb{Q}, +, -, *, \leq)$ where the multiplication $*$ is only linear.

Fourier-Motzkin elimination Given an LRA formula, its satisfiability can be decided as follows: Fourier-Motzkin elimination

1. Replace:
   - $(t_1 \neq t_2)$ with $(t_1 < t_2) \vee (t_2 < t_1)$.
   - $(t_1 \leq t_2)$ with $(t_1 < t_2) \vee (t_1 = t_2)$.
2. Eliminate equalities and apply the Fourier-Motzkin elimination[1] method on all variables to determine satisfiability.

Remark. Not practical on a large number of constraints. The simplex algorithm is more suited.

Linear integer arithmetic LIA theory has signature $\Sigma_{\text{LRA}} = (\mathbb{Z}, +, -, *, \leq)$ where the multiplication $*$ is only linear.

Fourier-Motzkin can be applied to check satisfiability. Simplex and branch & bound is usually better.

[1] `https://en.wikipedia.org/wiki/Fourier%E2%80%93Motzkin_elimination`

1.4.3 Difference logic theory

Difference logic (DL) atomic formulas have form $(x - y \leq k)$ where x, y are variables and k is a constant.

Remark. Constraints with form $(x - y \bowtie k)$ where $\bowtie \in \{=, \neq, <, \geq, >\}$ can be rewritten using $\leq$.

Remark. Unary constraints $x \leq k$ can be rewritten as $x - z_0 \leq k$ with the assignment $z_0 = 0$.

Theorem 1.4.1. By allowing $\neq$ and with domain in $\mathbb{Z}$, deciding satisfiability becomes NP-hard.

Proof. Any graph k-coloring instance can be poly-reduced to a difference logic formula. □

Graph consistency

Graph consistency Given DL literals Φ, it is possible to build a weighted graph $\mathcal{G}_\Phi$ where:

Nodes Variables occurring in Φ.

Edges $x \xrightarrow{k} y$ for each $(x - y \leq k) \in \Phi$.

Theorem 1.4.2. Φ is inconsistent $\iff$ $\mathcal{G}_\Phi$ has a negative cycle (i.e. cycle whose cost is negative).

Remark. A negative cycle acts as an inconsistency explanation (not necessarily minimal).

Remark. From the consistency graph, if there is a path from x to y with cost k, then $(x - y \leq k)$ can be deduced.

1.5 Combining theories

Given $\mathcal{T}_i$-solvers for theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$, a general approach to combine them into a $\bigcup_i^n \mathcal{T}_i$-solver is the following:

1. Purify the formula so that each literal belongs to a single theory. New constants can be introduced.

   Interface equalities Equalities involving shared constants across solvers should be the same for all solvers.

2. Iteratively run the following:
   a) Each $\mathcal{T}_i$-solver checks the satisfiability of $\mathcal{T}_i$-formulas. If one detects unsatisfiability, the whole formula is unsatisfiable.
   b) When a $\mathcal{T}_i$-solver deduces a new literal, it sends it to the other solvers.

Example. Consider the formula:

$$\Big(f\big(f(x) - f(y)\big) = a\Big) \wedge \Big(f(a) = a + 2\Big) \wedge \Big(x = y\Big)$$

where the theories of EUF and linear arithmetic (LA) are involved.
To determine satisfiability, the following steps are taken:

1. The formula is purified to obtain the literals:

LA	EUF
$e_1 = e_2 - e_3$ $e_4 = 0$ $e_5 = a + 2$	$f(e_1) = a$ $e_2 = f(x)$ $e_3 = f(y)$ $f(e_4) = e_5$ $x = y$

where $e_1, \ldots, e_5$ are new constants.

2. Both EUF-solver and LA-solver determine `SAT`. Moreover, the EUF-solver deduces that $\{x = y, f(x) = e_2, f(y) = e_3\} \models_{EUF} (e_2 = e_3)$ and sends it to the LA-solver.

LA	EUF
$e_1 = e_2 - e_3$ $e_4 = 0$ $e_5 = a + 2$ $\underline{e_2 = e_3}$	$f(e_1) = a$ $e_2 = f(x)$ $e_3 = f(y)$ $f(e_4) = e_5$ $x = y$

3. Both EUF-solver and LA-solver determine `SAT`. Moreover, the LA-solver deduces that $\{e_2 - e_3 = e_1, e_4 = 0, e_2 = e_3\} \models_{LA} (e_1 = e_4)$ and sends it to the EUF-solver.

LA	EUF
$e_1 = e_2 - e_3$ $e_4 = 0$ $e_5 = a + 2$ $e_2 = e_3$	$f(e_1) = a$ $e_2 = f(x)$ $e_3 = f(y)$ $f(e_4) = e_5$ $x = y$ $\underline{e_1 = e_4}$

$\vdots$

4. The EUF-solver determines `SAT` but the LA-solver determines `UNSAT`. Therefore, the formula is unsatisfiable.

1.5.1 Deterministic Nelson-Oppen

Deterministic Nelson-Oppen

Let $\mathcal{T}_1$ be a Σ_1-theory and $\mathcal{T}_2$ be a Σ_2-theory. $(\mathcal{T}_1 \cup \mathcal{T}_2)$-satisfiability can be checked with the deterministic Nelson-Oppen if $\mathcal{T}_1$ and $\mathcal{T}_2$ are:

Signature-disjoint $\Sigma_1 \cap \Sigma_2 = \varnothing$.

Stably-infinite $\mathcal{T}_i$ is stably-infinite iff every $\mathcal{T}_i$-satisfiable formula φ has a corresponding $\mathcal{T}_i$-model with a universe of infinite cardinality that satisfies it.

Convex For each set of $\mathcal{T}_i$-literals S, it holds that:

$$\big(S \models_{\mathcal{T}_i} (a_1 = b_1) \vee \cdots \vee (a_n = b_n)\big) \Rightarrow \big(S \models_{\mathcal{T}_i} (a_k = b_k)\big) \text{ for some } k \in \{1, \ldots, n\}$$

Example. $\mathcal{T}_{\mathbb{Z}}$ is not convex. Consider the following formula φ:

$$(1 \leq z) \wedge (z \leq 2) \wedge (u = 1) \wedge (v = 2)$$

We have that:

$$\varphi \models_{\mathcal{T}_{\mathbb{Z}}} (z = u) \vee (z = v)$$

But it is not true that:

$$\varphi \not\models_{\mathcal{T}_{\mathbb{Z}}} z = u \qquad \varphi \not\models_{\mathcal{T}_{\mathbb{Z}}} z = v$$

Algorithm Given a $(\mathcal{T}_1 \cup \mathcal{T}_2)$-formula S, the deterministic Nelson-Oppen algorithm works as follows:

1. Purify S into S_1 and S_2. Let E be the set of interface equalities between S_1 and S_2 (i.e. it contains all the equalities that involve shared constants).
2. If $S_1 \models_{\mathcal{T}_1} \bot$ or $S_2 \models_{\mathcal{T}_2} \bot$, then S is unsatisfiable.
3. If $S_1 \models_{\mathcal{T}_1} (x = y)$ with $(x = y) \in (E \setminus S_2)$, then $S_2 \leftarrow S_2 \cup \{x = y\}$. Go to Point 2.
4. If $S_2 \models_{\mathcal{T}_2} (x = y)$ with $(x = y) \in (E \setminus S_1)$, then $S_1 \leftarrow S_1 \cup \{x = y\}$. Go to Point 2.
5. S is satisfiable.

1.5.2 Non-deterministic Nelson-Oppen

Non-deterministic Nelson-Oppen

Extension of the deterministic Nelson-Oppen algorithm to non-convex theories.
Works by doing case splitting on pairs of shared variables and has exponential time complexity.

1.6 SMT extensions

1.6.1 Layered solvers

Layered solvers

Stratify the problem into layers of increasing complexity. The satisfiability of each layer is determined by a different solver of increasing expressivity and complexity.

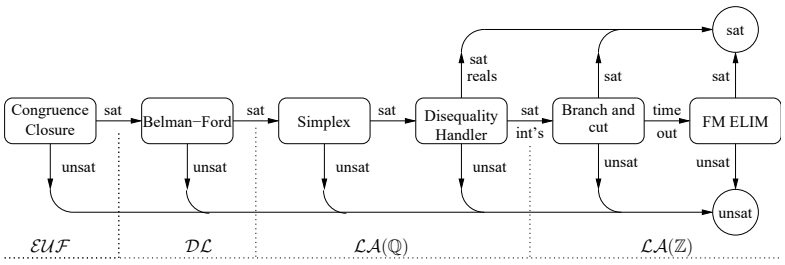


Figure 1.1: Example of layered solvers

1.6.2 Case splitting

Case splitting

Case reasoning on free variables.

Example. Given the formula:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big)$$

A solver can explore the case when $i = j$ and $i \neq j$.

$\mathcal{T}$-solver case reasoning The $\mathcal{T}$solver internally detects inconsistencies through case reasoning.

SAT solver case reasoning The $\mathcal{T}$-solver encodes the case reasoning and sends it to the SAT solver.

Example. Given the formula:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big)$$

The $\mathcal{T}$-solver sends to the SAT solver the following:

$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big) \wedge (i = j) \Rightarrow y = x$$
$$y = \texttt{read}\big(\texttt{write}(A, i, x), j\big) \wedge (i \neq j) \Rightarrow y = \texttt{read}(A, j)$$

1.6.3 Optimization modulo theory

Optimization modulo theory

Extension of SMT so that it finds a model that simultaneously satisfies the input formula φ and optimizes an objective function f_{obj}.
φ belongs to a theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ where $\mathcal{T}_{\preceq}$ contains a predicate $\preceq$ (e.g. $\leq$) representing a total order.

Offline OTM($\mathcal{LRA}$) Approach that does not require to modify the SMT solver.

- **Linear search** Repeatedly solve the problem and, at each iteration, add the constraint $\texttt{cost} < c_i$ where c_i is the cost found at the i-th iteration.
- **Binary search** Given the cost domain $[l_i, u_i]$, repeatedly pick a pivot $p_i \in [l_i, u_i]$ and add the constraint $\texttt{cost} < p_i$. If a model is found, recurse in the domain $[l_i, p_i]$, otherwise recurse in $[p_i, u_i]$.

Inline OTM($\mathcal{LRA}$) SMT solver that integrates an optimization procedure.

2 Linear programming

Linear programming (LP) Optimization problem defined through a system of linear constraints (equalities and inequalities) and an objective function. Linear programming (LP)

In the Cartesian plane, equalities represent hyperplanes and inequalities are half-spaces.

Remark. LP is useful for optimal allocation with limited number of resources.

Feasible solution Assignment satisfying all the constraints. Feasible solution

In the Cartesian plane, it is represented by any point within the intersection of all half-spaces defined by the inequalities.

Feasible region Set of all feasible solutions. It is a convex polyhedron that can be empty, bounded or unbounded. Feasible region

Remark. The optimal solution is always at one of the intersection points of the constraints within the feasible region (i.e. vertexes of the polyhedron).

The number of vertexes is finite but might grow exponentially.

Canonical form A linear programming problem is in canonical form if it is defined as: Canonical form

$$\max \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} a_{i,j} x_j \leq b_i \text{ for } 1 \leq i \leq m \wedge$$
$$x_j \geq 0 \text{ for } 1 \leq j \leq n$$

where:

- m is the number of constraints.
- n is the number of non-negative variables.
- $a_{i,j}, b_i, c_j \in \mathbb{R}$ are known parameters (given by the definition of the problem).
- $\sum_{j=1}^{n} c_j x_j$ is the objective function to maximize.
- $\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$ are m linear inequalities.

In matrix form:

$$\max\{\mathbf{cx}\} \text{ subject to } \boldsymbol{A}\mathbf{x} \leq \mathbf{b} \wedge \mathbf{x} \geq \bar{\mathbf{0}}$$

where:

- $\mathbf{c} = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$.
- $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$.
- $\mathbf{b} = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix}^T$.
- $\boldsymbol{A} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix}$

Standard form A linear programming problem is in standard form if it only has equality constraints (excluded those on single variables):

Standard form

$$\max \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} a_{i,j} x_j = b_i \text{ for } 1 \leq i \leq m \wedge$$
$$x_j \geq 0 \text{ for } 1 \leq j \leq n$$

In matrix form: $\max\{\mathbf{cx}\}$ subject to $\boldsymbol{A}\mathbf{x} = \mathbf{b} \wedge \mathbf{x} \geq \bar{\mathbf{0}}$.

Remark (Canonical to standard form)**.** Any LP problem with m constraints in canonical form has an equivalent standard form with m slack variables $y_1, \ldots, y_m \geq 0$ such that:

$$\forall i \in \{1, \ldots, m\} : \left(\sum_{j=1}^{n} a_{i,j} x_j \leq b_i \right) \Rightarrow \left(\sum_{j=1}^{n} a_{i,j} x_j + y_i = b_i \wedge y_i \geq 0 \right)$$

Example (Brewery problem)**.** The definition of the problem in canonical form is the following:

$$\begin{array}{lrcrcr} \max & 13A & + & 23B & & \\ \text{subj. to} & 5A & + & 15B & \leq & 480 \\ & 4A & + & 4B & \leq & 160 \\ & 35A & + & 20B & \leq & 1190 \\ & A & , & B & \geq & 0 \end{array}$$

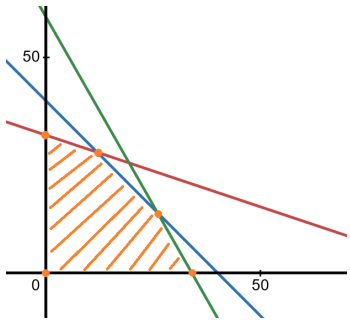


Figure 2.1: Feasible regions and vertexes of the polyhedron

In standard form, it becomes:

$$\begin{array}{lrcrcrcrcrcr} \max & 13A & + & 23B & & & & & & & & \\ \text{subj. to} & 5A & + & 15B & + & S_C & & & & & = & 480 \\ & 4A & + & 4B & & & + & S_H & & & = & 160 \\ & 35A & + & 20B & & & & & + & S_M & = & 1190 \\ & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0 \end{array}$$

2.1 Simplex algorithm

Algorithm that starts from an extreme point of the polyhedron and iteratively moves to a neighboring vertex as long as the objective function does not decrease.

2.1.1 Basis

Basis Given an LP problem $\mathcal{P}$ in standard form with m constraints and n variables (note: in standard form, it holds that $m \leq n$) and its constraint matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, a (ordered) basis $\mathcal{B} = \{x_{i_1}, \ldots, x_{i_m}\}$ is a subset of m of the n variables such that the columns $\boldsymbol{A}_{i_1}, \ldots, \boldsymbol{A}_{i_m}$ of $\boldsymbol{A}$ form a $m \times m$ invertible matrix $\boldsymbol{A}_{\mathcal{B}}$. Basis

Variables in $\mathcal{B}$ are basic variables while $\mathcal{N} = \{x_1, \ldots, x_n\} \setminus \mathcal{B}$ are non-basic variables.

$\mathcal{P}$ can be rewritten by separating basic and non-basic variables:

$$\max\{\mathbf{c}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} + \mathbf{c}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}\} \text{ subject to } \boldsymbol{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} + \boldsymbol{A}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}} = \mathbf{b} \wedge \mathbf{x}_{\mathcal{B}}, \mathbf{x}_{\mathcal{N}} \geq \bar{\mathbf{0}}$$

Basic solution By constraining $\mathbf{x}_{\mathcal{N}} = \bar{\mathbf{0}}$, an LP problem becomes: Basic solution

$$\max\{\mathbf{c}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}}\} \text{ subject to } \boldsymbol{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b} \wedge \mathbf{x}_{\mathcal{B}} \geq \bar{\mathbf{0}}$$

As $\boldsymbol{A}_{\mathcal{B}}$ is invertible by definition, it holds that:

$$\mathbf{x}_{\mathcal{B}} = \boldsymbol{A}_{\mathcal{B}}^{-1}\mathbf{b} \qquad \text{and} \qquad \max\{\mathbf{c}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}}\} = \max\{\mathbf{c}_{\mathcal{B}}\boldsymbol{A}_{\mathcal{B}}^{-1}\mathbf{b}\}$$

$\mathbf{x}_{\mathcal{B}}$ is a basic solution for $\mathcal{B}$.

Basic feasible solution (BFS) Given a basic solution $\mathbf{x}_{\mathcal{B}}$ for $\mathcal{B}$, it is feasible iff: Basic feasible solution (BFS)

$$\forall_{i=1}^{m} \mathbf{x}_{\mathcal{B}_i} \geq 0$$

Non-degenerate BFS A basic feasible solution is non-degenerate iff $\forall_{i=1}^{m} \mathbf{x}_{\mathcal{B}_i} > 0$. Non-degenerate BFS

Remark. A non-degenerate BFS is represented by a unique basis.

Remark. The simplex algorithm iteratively moves through basic feasible solutions $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots$ such that $\mathbf{c}\tilde{\mathbf{x}}_k \geq \mathbf{c}\tilde{\mathbf{x}}_{k-1}$.

2.1.2 Tableau

Tabular representation to describe the steps of the simplex algorithm. A variable Z is introduced to represent the value of the objective function (which can be seen as a conversion of the objective function into a constraint). Tableau
The tableau of an LP problem in standard form is divided into three sections:

1. Objective function, where the coefficients of the variables are called reduced costs. Reduced costs
2. Equality constraints.
3. Variable constraints.

Example (Brewery problem)**.** In standard form, the brewery problem is defined as:

$$\begin{array}{lrcrcrcrcrcr}
\max & 13A & + & 23B & & & & & & & & \\
\text{subj. to} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_H & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_M & = & 1190 \\
 & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}$$

As a tableau, assuming an initial basis $\mathcal{B} = \{S_C, S_H, S_M\}$, the problem is represented as:

$13A$	$+$	$23B$							$-$	Z	$=$	0
$5A$	$+$	$15B$	$+$	S_C							$=$	480
$4A$	$+$	$4B$			$+$	S_H					$=$	160
$35A$	$+$	$20B$					$+$	S_M			$=$	1190
A	$,$	B	$,$	S_C	$,$	S_H	$,$	S_M			$\geq$	0

The reduced costs are $\{13, 23, 0, 0, 0\}$.

2.1.3 Pivoting

Given a basis $\mathcal{B}$, it is possible to insert a new variable $x^{\text{in}} \notin \mathcal{B}$ into it and remove an old one $x^{\text{out}} \in \mathcal{B}$ to increase (or leave unchanged) the objective function:

Entering variable x^{in} should be the variable in $\mathcal{N}$ with the highest improvement on the objective function. Entering variable

Leaving variable x^{out} should be chosen to ensure that the new basis $\mathcal{B}' = \mathcal{B} \cup \{x^{\text{in}}\} \setminus \{x^{\text{out}}\}$ is still a feasible basis. Leaving variable

Minimum ratio rule For each constraint i (i.e. i-th row of the system $\boldsymbol{A}\mathbf{x} = \mathbf{b}$), it is possible to compute the ratio: Minimum ratio rule

$$\frac{\mathbf{b}_i}{\boldsymbol{\alpha}_i^{\text{in}}}$$

where:

- $\boldsymbol{\alpha}_i^{\text{in}}$ is the coefficient associated to the entering variable x^{in} in the i-th constraint.
- $\mathbf{b}_i$ is the known term of the i-th constraint.

The index i of the leaving variable $x^{\text{out}} = \mathcal{B}_i$ is determined as:

$$\arg\min_i \frac{\mathbf{b}_i}{\boldsymbol{\alpha}_i^{\text{in}}}$$

Once x^{in} and $x^{\text{out}} = \mathcal{B}_i$ has been determined, x^{in} is isolated in the equation of the i-th constraint and it is substituted in all the others.

Example (Brewery problem)**.** The initial tableau of the brewery problem with $\mathcal{B} = \{S_C, S_H, S_M\}$ and $\mathcal{N} = \{A, B\}$ is:

$13A$	$+$	$23B$							$-$	Z	$=$	0
$5A$	$+$	$15B$	$+$	S_C							$=$	480
$4A$	$+$	$4B$			$+$	S_H					$=$	160
$35A$	$+$	$20B$					$+$	S_M			$=$	1190
A	$,$	B	$,$	S_C	$,$	S_H	$,$	S_M			$\geq$	0

1. It can be easily seen that $x^{\text{in}} = B$ should be the entering variable.

   For the leaving variable, the ratios are:

$$\arg\min \left\{ \frac{480}{15}, \frac{160}{4}, \frac{1190}{20} \right\} = \arg\min \{32, 40, 59.5\} = 1$$

Therefore, the leaving variable is $x^{\text{out}} = \mathcal{B}_1 = S_C$.

We now isolate B from the first constraint:

$$5A + 15B + S_C = 480 \iff \frac{1}{3}A + B + \frac{1}{15}S_C = 32$$
$$\iff B = 32 - \frac{1}{3}A - \frac{1}{15}S_C$$

The tableau with $\mathcal{B}' = \{B, S_H, S_M\}$ and $\mathcal{N}' = \{A, S_C\}$ is updated as:

$\frac{16}{3}A$			$-$	$\frac{23}{15}S_C$					$-$	Z	$=$	-736
$\frac{1}{3}A$	$+$	B	$+$	$\frac{1}{15}S_C$							$=$	32
$\frac{8}{3}A$			$-$	$\frac{4}{15}S_C$	$+$	S_H					$=$	32
$\frac{85}{3}A$			$-$	$\frac{4}{3}S_C$			$+$	S_M			$=$	550
A	,	B	,	S_C	,	S_H	,	S_M			$\geq$	0

2. Now, $x^{\text{in}} = A$ is the variable that increases the objective function the most.

   For the leaving variable, the ratios are:

$$\arg\min\left\{\frac{32}{1/3}, \frac{32}{8/3}, \frac{550}{85/3}\right\} = 2$$

   Therefore, the leaving variable is $x^{\text{out}} = \mathcal{B}'_2 = S_H$.

   A isolated in the second constraint brings to:

$$A = \frac{3}{8}(32 + \frac{4}{15}S_C - S_H) \iff A = 12 + \frac{1}{10}S_C - \frac{3}{8}S_H$$

   The tableau with $\mathcal{B}'' = \{A, B, S_M\}$ and $\mathcal{N}'' = \{S_C, S_H\}$ is updated as:

		$-$	S_C	$-$	$2S_H$			$-$	Z	$=$	-800
	B	$+$	$\frac{1}{10}S_C$	$+$	$\frac{1}{8}S_H$					$=$	28
A		$-$	$\frac{1}{10}S_C$	$+$	$\frac{3}{8}S_H$					$=$	12
		$-$	$\frac{25}{6}S_C$	$-$	$\frac{85}{8}S_H$	$+$	S_M			$=$	210
A	, B	,	S_C	,	S_H	,	S_M			$\geq$	0

   We cannot continue anymore as the reduced costs $\{0, 0, -1, -2, 0\}$ are ≤ 0 (i.e. cannot improve the objective function anymore).

2.1.4 Optimality

Optimality

When any substitution worsens the objective function, the current assignment is optimal. In the tableau, this happens when all the reduced costs are ≤ 0.

Remark. For any optimal solution, there is at least a basis such that the reduced costs are ≤ 0. Therefore, this is a sufficient condition.

Remark. The fact that reduced costs are ≤ 0 is not a necessary condition.

Example (Brewery problem)**.** The tableau at the last iteration, with $\mathcal{B}'' = \{A, B, S_M\}$ and $\mathcal{N}'' = \{S_C, S_H\}$, is the following:

			$-$	S_C	$-$	$2S_H$			$-$	Z	$=$	-800
		B	$+$	$\frac{1}{10}S_C$	$+$	$\frac{1}{8}S_H$					$=$	28
A			$-$	$\frac{1}{10}S_C$	$+$	$\frac{3}{8}S_H$					$=$	12
			$-$	$\frac{25}{6}S_C$	$-$	$\frac{85}{8}S_H$	$+$	S_M			$=$	210
A	,	B	,	S_C	,	S_H	,	S_M			$\geq$	0

All reduced costs are ≤ 0 and an optimal solution has been reached:

- $S_C = 0$ and $S_H = 0$ (variables in $\mathcal{N}''$).
- $A = 12$, $B = 28$ and $S_M = 210$ (variables in $\mathcal{B}''$. Obtained by isolating them in the constraints).
- $-S_C - 2S_H - Z = -800 \iff Z = 800 - S_C - 2S_H \iff Z = 800$ (objective function).

Remark. The points in the feasible region of a problem $\mathcal{P}$ are:

$$\mathcal{F}_\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{A}\mathbf{x} \leq \mathbf{b} \wedge \mathbf{x} \geq \bar{\mathbf{0}}\}$$

Optimal region For an LP problem $\mathcal{P}$, its set of solutions is defined as: Optimal region

$$\mathcal{O}_\mathcal{P} = \{x^* \in \mathcal{F}_\mathcal{P} \mid \forall \mathbf{x} \in \mathcal{F}_\mathcal{P} : \mathbf{c}\mathbf{x}^* \geq \mathbf{c}\mathbf{x}\}$$

Trivially, it holds that $\mathcal{O}_\mathcal{P} \subseteq \mathcal{F}_\mathcal{P}$ and $\mathcal{F}_\mathcal{P} = \varnothing \Rightarrow \mathcal{O}_\mathcal{P} = \varnothing$.

Theorem 2.1.1. If $\mathcal{O}_\mathcal{P}$ is finite, then $|\mathcal{O}_\mathcal{P}| = 1$ (therefore, if $|\mathcal{O}_\mathcal{P}| > 1$, then $\mathcal{O}_\mathcal{P}$ is infinite).

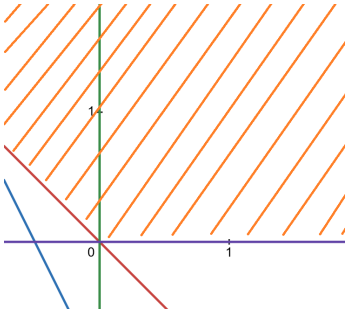
Unbounded problem An LP problem $\mathcal{P}$ is unbounded if it does not have an optimal solution (i.e. $\mathcal{F}_\mathcal{P}$ is an unbounded polyhedron). Unbounded problem

With the tableau formulation, if a column has reduced cost > 0 and all the constraint coefficients are ≤ 0, then the problem is unbounded.

Example. Given the following tableau:

	x	$-$	y					$-$	Z	$=$	-1
$-$	x	$-$	y	$+$	S_1					$=$	0
$-$	$2x$	$-$	y			$+$	S_2			$=$	1
	x	,	y	,	S_1	,	S_2			$\geq$	0

The unboundedness of the problem can be detected from the first column.



2.1.5 Algorithm

Given an LP problem $\mathcal{P}$ in standard form, the steps of the simplex algorithm are:

1. Set $k = 0$, find a feasible basis $\mathcal{B}_k$ and reformulate $\mathcal{P}$ according to it.
2. If the basis feasible solution is optimal, return.
3. If $\mathcal{P}$ is unbounded, return.
4. Select an entering variable x^{in}.
5. Select a leaving variable x^{out}.
6. Let $\mathcal{B}_{k+1} = \mathcal{B}_k \cup \{x^{\text{in}}\} \setminus \{x^{\text{out}}\}$ and reformulate $\mathcal{P}$ according to the new basis.
7. Set $k = k + 1$ and go to Point 2.

Properties

- If all basis feasible solutions are non-degenerate, the simplex algorithm always terminates (as the solution is always strictly improving).
- In the general case, the algorithm might stall by ending up in a loop.
- The worst-case time complexity is $O(2^n)$. The average case is polynomial.

Remark. If the problem has lots of vertexes, the interior point method (polynomial complexity) or approximation algorithms should be preferred.

2.1.6 Two-phase method

Two-phase method

Method that solves an LP problem by first finding an initial feasible basis $\mathcal{B}_0$ and determining if the LP problem is unsatisfiable.
Given an LP problem $\mathcal{P}$ ($\max\{\mathbf{cx}\}$ subject to $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ with m constraints and n variables), the two-phase method works as follows:

Phase 1 Define a new artificial problem $\mathcal{P}'$ from $\mathcal{P}$ with new variables $s_1, \dots, s_m$ as follows:

$$\max\left\{-\sum_{i=1}^{m} s_i\right\} \text{ subject to } \begin{aligned} &\sum_{j=1}^{n} a_{i,j}x_j + s_i = b_i \text{ for } i \in \{k \in \{1,\dots,m\} \mid b_k \geq 0\} \wedge \\ &\sum_{j=1}^{n} a_{i,j}x_j - s_i = b_i \text{ for } i \in \{k \in \{1,\dots,m\} \mid b_k < 0\} \wedge \\ &s_i, x_j \geq 0 \end{aligned}$$

Remark. It holds that $-\sum_{i=1}^{m} s_i \leq 0$ and $\mathcal{B}' = \{s_1, \dots, s_m\}$ is always a feasible basis corresponding to the basis feasible solution $x_j = 0$, $s_i = |b_i|$

The problem $\mathcal{P}'$ with basis $\mathcal{B}'$ can be solved through the simplex method.

Theorem 2.1.2. Let $\mathcal{F}_\mathcal{P}$ be the feasible region of $\mathcal{P}$. It holds that:

$$\mathcal{F}_\mathcal{P} \neq \varnothing \iff \sum_{i=1}^{m} s_i = 0$$

In other words:

- If the optimal solution of $\mathcal{P}'$ is < 0, then $\mathcal{P}$ is unsatisfiable.
- Otherwise, the basis $\mathcal{B}_{\mathcal{P}'}$ corresponding to the optimal solution of $\mathcal{P}'$ can be used as the initial basis of $\mathcal{P}$, after removing the variables s_i.

Phase 2 Solve $\mathcal{P}$ through the simplex algorithm using as initial basis $\mathcal{B}_{\mathcal{P}'}$.