Combinatorial Decision Making and Optimization (Module 2)

Last update: 11 April 2024

Academic Year 2023 – 2024
Alma Mater Studiorum · University of Bologna

Contents

1 Satisfiability modulo theory (SMT)

1.1 First-order logic for SMT

1.1.1 Syntax

Remark. Only quantifier-free fragments will be considered in this course.

Functions The set of all the functions is denoted as $\Sigma^F = \bigcup_{k \geq 0} \Sigma_k^F$ where Σ_k^F denotes the set of k-ary functions. Functions

Constants Σ_0^F

Predicates The set of all the predicates is denoted as $\Sigma^P = \bigcup_{k \geq 0} \Sigma_k^P$ where Σ_k^P denotes the set of k-ary predicates. Predicates

Propositional symbols Σ_0^P

Signature The set of the non-logical symbols of FOL is denoted as: Signature

$$\Sigma = \Sigma^F \cup \Sigma^P$$

Terms The set of terms over Σ is denoted as $\mathbb{T}^\Sigma$. Terms

Formulas The set of formulas over Σ is denoted as $\mathbb{F}^\Sigma$. Formulas

1.1.2 Semantics

Σ-model Pair $\mathcal{M} = \langle M, (\cdot)^\mathcal{M} \rangle$ defined on a given Σ where: Σ-model

- M is the universe of $\mathcal{M}$.
- $(\cdot)^\mathcal{M}$ is a mapping such that:
  - $\forall f \in \Sigma_k^F : f^\mathcal{M} \in \{\varphi \mid \varphi : M^k \to M\}$.
  - $\forall p \in \Sigma_k^P : p^\mathcal{M} \in \{\varphi \mid \varphi : M^k \to \{\texttt{true}, \texttt{false}\}\}$.

Interpretation Extension of the mapping function $(\cdot)^\mathcal{M}$ to terms and formulas: Interpretation

- $\top^\mathcal{M} = \texttt{true}$ and $\bot^\mathcal{M} = \texttt{false}$.
- $(f(t_1, \dots, t_k))^\mathcal{M} = f^\mathcal{M}(t_1^\mathcal{M}, \dots, t_k^\mathcal{M})$ and $(p(t_1, \dots, t_k))^\mathcal{M} = p^\mathcal{M}(t_1^\mathcal{M}, \dots, t_k^\mathcal{M})$.
- $\texttt{ite}(\varphi, t_1, t_2)^\mathcal{M} = \begin{cases} t_1^\mathcal{M} & \text{if } \varphi^\mathcal{M} = \texttt{true} \\ t_2^\mathcal{M} & \text{if } \varphi^\mathcal{M} = \texttt{false} \end{cases}$

Remark. `ite` is an auxiliary function to capture the if-else construct.

1.1.3 Σ-theory

Satisfiability A model $\mathcal{M}$ satisfies a formula $\varphi \in \mathcal{F}^\Sigma$ if $\varphi^\mathcal{M} = \texttt{true}$. Satisfiability

Σ-theory Possibly infinite set $\mathcal{T}$ of Σ-models. Σ-theory

$\mathcal{T}$-satisfiability A formula $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-satisfiable if there exists a model $\mathcal{M} \in \mathcal{T}$ that satisfies it. $\mathcal{T}$-satisfiability

$\mathcal{T}$-consistency A set of formulas $\{\varphi_1, \dots, \varphi_k\} \subseteq \mathbb{F}^\Sigma$ is $\mathcal{T}$-consistent iff $\varphi_1 \wedge \cdots \wedge \varphi_k$ is $\mathcal{T}$-satisfiable. $\mathcal{T}$-consistency

$\mathcal{T}$-entailment A set of formulas $\Gamma \subseteq \mathbb{F}^\Sigma$ $\mathcal{T}$-entails a formula $\varphi \in \mathbb{F}^\Sigma$ ($\Gamma \models_\mathcal{T} \varphi$) iff in every model $\mathcal{M} \in \mathcal{T}$ that satisfies Γ, φ is also satisfied. $\mathcal{T}$-entailment

Remark. Γ is $\mathcal{T}$-consistent iff $\Gamma \not\models_\mathcal{T} \bot$.

$\mathcal{T}$-validity A formula $\varphi \in \mathbb{F}^\Sigma$ is $\mathcal{T}$-valid iff $\varnothing \models_\mathcal{T} \varphi$. $\mathcal{T}$-validity

Remark. φ is $\mathcal{T}$-consistent iff $\neg\varphi$ is not $\mathcal{T}$-valid.

Theory lemma $\mathcal{T}$-valid clause $c = l_1 \vee \cdots \vee l_k$. Theory lemma

$\mathcal{T}$-expansion Given a Σ-model $\mathcal{M} = \langle M, (\cdot)^\mathcal{M} \rangle$ and $\Sigma' \supseteq \Sigma$, an expansion $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ to Σ' is any Σ'-model such that: $\mathcal{T}$-expansion

- $M' = M$.
- $\forall s \in \Sigma : s^{\mathcal{M}'} = s^\mathcal{M}$

Remark. Given a Σ-theory $\mathcal{T}$, we implicitly consider the theory $\mathcal{T}'$ as:

$$\mathcal{T}' = \{\mathcal{M}' \mid \mathcal{M}' \text{ is an expansion of a } \Sigma\text{-model } \mathcal{M}\}$$

Ground $\mathcal{T}$-satisfiability Given a Σ-theory $\mathcal{T}$, determine if a ground formula is $\mathcal{T}$-satisfiable over a Σ-expansion $\mathcal{T}'$. Ground $\mathcal{T}$-satisfiability

Axiomatically defined theory Given a minimal set of formulas (axioms) $\Lambda \subseteq \mathbb{F}^\Sigma$, its corresponding theory is the set of all the models of Λ. Axiomatically defined theory

Example. Let Σ be defined as:

$$\Sigma_0^F = \{a, b, c, d\} \qquad \Sigma_1^F = \{f, g\} \qquad \Sigma_2^P = \{p\}$$

A Σ-model $\mathcal{M} = \langle [0, 2\pi[, (\cdot)^\mathcal{M} \rangle$ can be defined as follows:

$$a^\mathcal{M} = 0 \qquad b^\mathcal{M} = \frac{\pi}{2} \qquad c^\mathcal{M} = \pi \qquad d^\mathcal{M} = \frac{3\pi}{2}$$

$$f^\mathcal{M} = \sin \qquad g^\mathcal{M} = \cos \qquad p^\mathcal{M}(x, y) \iff x > y$$

To determine if $p(g(x), f(d))$ is $\mathcal{M}$-satisfiable, we have to expand $\mathcal{M}$. Let $\Sigma' = \Sigma \cup \{x\}$. The expansion $\mathcal{M}'$ such that $x^{\mathcal{M}'} = \frac{\pi}{2}$ makes the formula satisfiable.