Distributed Autonomous Systems

Last update: 25 February 2025

Academic Year 2024 – 2025
Alma Mater Studiorum · University of Bologna

Contents

1 Averaging systems

1.1 Graphs

1.1.1 Definitions

Directed graph (digraph) Pair $G = (I, E)$ where $I = \{1, \dots, N\}$ is the set of nodes and $E \subseteq I \times I$ is the set of edges. Directed graph

Undirected graph Digraph where $\forall i, j : (i, j) \in E \Rightarrow (j, i) \in E$. Undirected graph

Subgraph Given a graph (I, E), (I', E') is a subgraph of it if $I' \subseteq I$ and $E' \subset E$. Subgraph

Spanning subgraph Subgraph where $I' = I$.

In-neighbor A node $j \in I$ is an in-neighbor of $i \in I$ if $(j, i) \in E$. In-neighbor

Set of in-neighbors The set of in-neighbors of $i \in I$ is the set: Set of in-neighbors

$$\mathcal{N}_i^{\text{IN}} = \{j \in I \mid (j, i) \in E\}$$

In-degree Number of in-neighbors of a node $i \in I$: In-degree

$$\deg_i^{\text{IN}} = |\mathcal{N}_i^{\text{IN}}|$$

Out-neighbor A node $j \in I$ is an out-neighbor of $i \in I$ if $(i, j) \in E$. Out-neighbor

Set of out-neighbors The set of out-neighbors of $i \in I$ is the set: Set of in-neighbors

$$\mathcal{N}_i^{\text{OUT}} = \{j \in I \mid (i, j) \in E\}$$

Out-degree Number of out-neighbors of a node $i \in I$: Out-degree

$$\deg_i^{\text{OUT}} = |\mathcal{N}_i^{\text{OUT}}|$$

Balanced digraph A digraph is balanced if $\forall i \in I : \deg_i^{\text{IN}} = \deg_i^{\text{OUT}}$. Balanced digraph

Periodic graph Graph where there exists a period $k > 1$ that divides the length of any cycle. Periodic graph

> **Remark.** A graph with self-loops is aperiodic.

Strongly connected digraph Digraph where each node is reachable from any node. Strongly connected digraph

Connected undirected graph Undirected graph where each node is reachable from any node. Connected undirected graph

Weakly connected digraph Digraph where its undirected version is connected. Weakly connected digraph

1.1.2 Weighted digraphs

Weighted digraph Triplet $G = (I, E, \{a_{i,j}\}_{(i,j)\in E})$ where (I, E) is a digraph and $a_{i,j} > 0$ is a weight for the edge (i, j). Weighted digraph

Weighted in-degree Sum of the weights of the inward edges: Weighted in-degree

$$\deg_i^{\text{IN}} = \sum_{j=1}^{N} a_{j,i}$$

Weighted out-degree Sum of the weights of the outward edges: Weighted out-degree

$$\deg_i^{\text{OUT}} = \sum_{j=1}^{N} a_{i,j}$$

Weighted adjacency matrix Non-negative matrix $\boldsymbol{A}$ such that $\boldsymbol{A}_{i,j} = a_{i,j}$: Weighted adjacency matrix

$$\begin{cases} \boldsymbol{A}_{i,j} > 0 & \text{if } (i,j) \in E \\ \boldsymbol{A}_{i,j} = 0 & \text{otherwise} \end{cases}$$

In/out-degree matrix Matrix where the diagonal contains the in/out-degrees: In/out-degree matrix

$$\boldsymbol{D}^{\text{IN}} = \begin{bmatrix} \deg_1^{\text{IN}} & 0 & \cdots & 0 \\ 0 & \deg_2^{\text{IN}} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \deg_N^{\text{IN}} \end{bmatrix} \qquad \boldsymbol{D}^{\text{OUT}} = \begin{bmatrix} \deg_1^{\text{OUT}} & 0 & \cdots & 0 \\ 0 & \deg_2^{\text{OUT}} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \deg_N^{\text{OUT}} \end{bmatrix}$$

Remark. Given a digraph with adjacency matrix $\boldsymbol{A}$, its reverse digraph has adjacency matrix $\boldsymbol{A}^T$.

Remark. It holds that:

$$\boldsymbol{D}^{\text{IN}} = \text{diag}(\boldsymbol{A}^T\mathbf{1}) \quad \boldsymbol{D}^{\text{OUT}} = \text{diag}(\boldsymbol{A}\mathbf{1})$$

where $\mathbf{1}$ is a vector of ones.

Remark. A digraph is balanced iff $\boldsymbol{A}^T\mathbf{1} = \boldsymbol{A}\mathbf{1}$.