Distributed Autonomous Systems

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Contents

1 Averaging systems

1.1 Graphs

1.1.1 Definitions

Directed graph (digraph) Pair $G = (I, E)$ where $I = \{1, \dots, N\}$ is the set of nodes and $E \subseteq I \times I$ is the set of edges. Directed graph

Undirected graph Digraph where $\forall i, j : (i, j) \in E \Rightarrow (j, i) \in E$. Undirected graph

Subgraph Given a graph (I, E), (I', E') is a subgraph of it if $I' \subseteq I$ and $E' \subset E$. Subgraph

Spanning subgraph Subgraph where $I' = I$.

In-neighbor A node $j \in I$ is an in-neighbor of $i \in I$ if $(j, i) \in E$. In-neighbor

Set of in-neighbors The set of in-neighbors of $i \in I$ is the set: Set of in-neighbors

$$\mathcal{N}_i^{\text{IN}} = \{j \in I \mid (j, i) \in E\}$$

In-degree Number of in-neighbors of a node $i \in I$: In-degree

$$\deg_i^{\text{IN}} = |\mathcal{N}_i^{\text{IN}}|$$

Out-neighbor A node $j \in I$ is an out-neighbor of $i \in I$ if $(i, j) \in E$. Out-neighbor

Set of out-neighbors The set of out-neighbors of $i \in I$ is the set: Set of in-neighbors

$$\mathcal{N}_i^{\text{OUT}} = \{j \in I \mid (i, j) \in E\}$$

Out-degree Number of out-neighbors of a node $i \in I$: Out-degree

$$\deg_i^{\text{OUT}} = |\mathcal{N}_i^{\text{OUT}}|$$

Balanced digraph A digraph is balanced if $\forall i \in I : \deg_i^{\text{IN}} = \deg_i^{\text{OUT}}$. Balanced digraph

Periodic graph Graph where there exists a period $k > 1$ that divides the length of any cycle. Periodic graph

> **Remark.** A graph with self-loops is aperiodic.

Strongly connected digraph Digraph where each node is reachable from any node. Strongly connected digraph

Connected undirected graph Undirected graph where each node is reachable from any node. Connected undirected graph

Weakly connected digraph Digraph where its undirected version is connected. Weakly connected digraph

1.1.2 Weighted digraphs

Weighted digraph Triplet $G = (I, E, \{a_{i,j}\}_{(i,j)\in E})$ where (I, E) is a digraph and $a_{i,j} > 0$ is a weight for the edge (i, j). Weighted digraph

Weighted in-degree Sum of the weights of the inward edges: Weighted in-degree

$$\deg_i^{\text{IN}} = \sum_{j=1}^{N} a_{j,i}$$

Weighted out-degree Sum of the weights of the outward edges: Weighted out-degree

$$\deg_i^{\text{OUT}} = \sum_{j=1}^{N} a_{i,j}$$

Weighted adjacency matrix Non-negative matrix $\boldsymbol{A}$ such that $\boldsymbol{A}_{i,j} = a_{i,j}$: Weighted adjacency matrix

$$\begin{cases} \boldsymbol{A}_{i,j} > 0 & \text{if } (i,j) \in E \\ \boldsymbol{A}_{i,j} = 0 & \text{otherwise} \end{cases}$$

In/out-degree matrix Matrix where the diagonal contains the in/out-degrees: In/out-degree matrix

$$\boldsymbol{D}^{\text{IN}} = \begin{bmatrix} \deg_1^{\text{IN}} & 0 & \cdots & 0 \\ 0 & \deg_2^{\text{IN}} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \deg_N^{\text{IN}} \end{bmatrix} \qquad \boldsymbol{D}^{\text{OUT}} = \begin{bmatrix} \deg_1^{\text{OUT}} & 0 & \cdots & 0 \\ 0 & \deg_2^{\text{OUT}} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \deg_N^{\text{OUT}} \end{bmatrix}$$

Remark. Given a digraph with adjacency matrix $\boldsymbol{A}$, its reverse digraph has adjacency matrix $\boldsymbol{A}^T$.

Remark. It holds that:

$$\boldsymbol{D}^{\text{IN}} = \text{diag}(\boldsymbol{A}^T\boldsymbol{1}) \quad \boldsymbol{D}^{\text{OUT}} = \text{diag}(\boldsymbol{A}\boldsymbol{1})$$

where $\boldsymbol{1}$ is a vector of ones.

Remark. A digraph is balanced iff $\boldsymbol{A}^T\boldsymbol{1} = \boldsymbol{A}\boldsymbol{1}$.

1.1.3 Laplacian matrix

(Out-degree) Laplacian matrix Matrix $\boldsymbol{L}$ defined as: Laplacian matrix

$$\boldsymbol{L} = \boldsymbol{D}^{\text{OUT}} - \boldsymbol{A}$$

Remark. The vector $\boldsymbol{1}$ is always an eigenvector of $\boldsymbol{L}$ with eigenvalue 0:

$$\boldsymbol{L}\boldsymbol{1} = (\boldsymbol{D}^{\text{OUT}} - \boldsymbol{A})\boldsymbol{1} = \boldsymbol{D}^{\text{OUT}}\boldsymbol{1} - \boldsymbol{D}^{\text{OUT}}\boldsymbol{1} = 0$$

In-degree Laplacian matrix Matrix $\boldsymbol{L}^{\text{IN}}$ defined as: In-degree Laplacian matrix

$$\boldsymbol{L}^{\text{IN}} = \boldsymbol{D}^{\text{IN}} - \boldsymbol{A}^T$$

Remark. $\boldsymbol{L}^{\text{IN}}$ is the out-degree Laplacian of the reverse graph.

1.2 Distributed algorithm

Distributed algorithm Given a network of N agents that communicate according to a (fixed) digraph G (each agent receives messages from its in-neighbors), a distributed algorithm computes:

Distributed algorithm

$$x_i^{k+1} = \texttt{stf}_i(x_i^k, \{x_j^k\}_{j \in \mathcal{N}_i^{\text{IN}}}) \quad i \in \{1, \dots, N\}$$

where x_i^k is the state of agent i at time k and $\texttt{stf}_i$ is a local state transition function that depends on the current input states.

Remark. Out-neighbors can also be used.

Remark. If all nodes have a self-loop, the notation can be compacted as:

$$x_i^{k+1} = \texttt{stf}_i(\{x_j\}_{j \in \mathcal{N}_i^{\text{IN}}}) \quad \text{or} \quad x_i^{k+1} = \texttt{stf}_i(\{x_j\}_{j \in \mathcal{N}_i^{\text{OUT}}})$$

1.2.1 Discrete-time averaging algorithm

Linear averaging distributed algorithm (in-neighbors) Given the communication digraph with self-loops $G^{\text{comm}} = (I, E)$ (i.e., $(j, i) \in E$ indicates that j sends messages to i), a linear averaging distributed algorithm is defined as:

Linear averaging distributed algorithm (in-neighbors)

$$x_i^{k+1} = \sum_{j \in \mathcal{N}_i^{\text{IN}}} a_{ij} x_j^k \quad i \in \{1, \dots, N\}$$

where $a_{ij} > 0$ is the weight of the edge $(j, i) \in E$.

Linear time-invariant (LTI) autonomous system By defining $a_{ij} = 0$ for $(j, i) \notin E$, the formulation becomes:

Linear time-invariant (LTI) autonomous system

$$x_i^{k+1} = \sum_{j=1}^{N} a_{ij} x_j^k \quad i \in \{1, \dots, N\}$$

In matrix form, it becomes:

$$x^{k+1} = \boldsymbol{A}^T x^k$$

where $\boldsymbol{A}$ is the adjacency matrix of G^{comm}.

Remark. This model is inconsistent with respect to graph theory as weights are inverted (i.e., a_{ij} refers to the edge (j, i)).

Linear averaging distributed algorithm (out-neighbors) Given a fixed sensing digraph with self-loops $G^{\text{sens}} = (I, E)$ (i.e., $(i, j) \in E$ indicates that j sends messages to i), the algorithm is defined as:

Linear averaging distributed algorithm (out-neighbors)

$$x_i^{k+1} = \sum_{j \in \mathcal{N}_i^{\text{OUT}}} a_{ij} x_j^k = \sum_{j=1}^{N} a_{ij} x_j^k$$

In matrix form, it becomes:

$$x^{k+1} = \boldsymbol{A} x^k$$

where $\boldsymbol{A}$ is the weighted adjacency matrix of G^{sens}.

1.2.2 Stochastic matrices

Row stochastic Given a square matrix $\boldsymbol{A}$, it is row stochastic if its rows sum to 1: Row stochastic

$$\boldsymbol{A}\mathbf{1} = \mathbf{1}$$

Column stochastic Given a square matrix $\boldsymbol{A}$, it is column stochastic if its columns sum to 1: Column stochastic

$$\boldsymbol{A}^T\mathbf{1} = \mathbf{1}$$

Doubly stochastic Given a square matrix $\boldsymbol{A}$, it is doubly stochastic if it is both row and column stochastic. Doubly stochastic

Lemma 1.2.1. Given a digraph G with adjacency matrix $\boldsymbol{A}$, if G is strongly connected and aperiodic, and $\boldsymbol{A}$ is row stochastic, its eigenvalues are such that:

- $\lambda = 1$ is a simple eigenvalue (i.e., algebraic multiplicity of 1),
- All others μ are $|\mu| < 1$.

Remark. For the lemma to hold, it is necessary and sufficient that G contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.

Theorem 1.2.2 (Consensus)**.** Consider a discrete-time averaging system with digraph G and weighted adjacency matrix $\boldsymbol{A}$. Assume G strongly connected and aperiodic, and $\boldsymbol{A}$ row stochastic. Consensus
It holds that there exists a left eigenvector $\mathbf{w} \in \mathbb{R}^N$, $\mathbf{w} > 0$ such that the consensus converges to:

$$\lim_{k\to\infty} x^k = \mathbf{1}\frac{\mathbf{w}^T x^0}{\mathbf{w}^T\mathbf{1}} = \begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} \frac{\sum_{i=1}^N w_i x_i^0}{\sum_{i=1}^N w_i} = \begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} \sum_{i=1}^N \frac{w_i}{\sum_{j=1}^N w_j} x_i^0$$

where $\tilde{w}_i = \frac{w_i}{\sum_{i=j}^N w_j}$ are all normalized and sum to 1 (i.e., they produce a convex combination).
Moreover, if $\boldsymbol{A}$ is doubly stochastic (e.g., G weight balanced with positive weights), then it holds that the consensus is the average:

$$\lim_{k\to\infty} x^k = \mathbf{1}\frac{1}{N}\sum_{i=1}^N x_i^0$$

Sketch of proof. Let $\boldsymbol{T} = \begin{bmatrix}\mathbf{1} & \mathbf{v}^2 & \cdots & \mathbf{v}^N\end{bmatrix}$ be a change in coordinates that transforms an adjacency matrix into its Jordan form $\boldsymbol{J}$:

$$\boldsymbol{J} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$$

As $\lambda = 1$ is a simple eigenvalue, it holds that:

$$\boldsymbol{J} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{J}_2 & \\ 0 & & & \end{bmatrix}$$

where the eigenvalues of $\boldsymbol{J}_2 \in \mathbb{R}^{(N-1)\times(N-1)}$ lie inside the open unit disk.
Let $x^k = \boldsymbol{T}\bar{x}^k$, then we have that:

$$\begin{aligned}
x^{k+1} &= \boldsymbol{A}x^k \iff \\
\boldsymbol{T}\bar{x}^{k+1} &= \boldsymbol{A}(\boldsymbol{T}\bar{x}^k) \iff \\
\bar{x}^{k+1} &= \boldsymbol{T}^{-1}\boldsymbol{A}(\boldsymbol{T}\bar{x}^k) = \boldsymbol{J}\bar{x}^k
\end{aligned}$$

Therefore:

$$\begin{gathered}
\lim_{k\to\infty} \bar{x}^k = \bar{x}_1^0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\bar{x}_1^{k+1} = \bar{x}_1^k \quad \forall k \geq 0 \\
\lim_{k\to\infty} \bar{x}_i^k = 0 \quad \forall i = 2, \dots, N
\end{gathered}$$

□

Example (Metropolis-Hasting weights)**.** Given an undirected unweighted graph G with edges of degrees $d_1, \dots, d_n$, Metropolis-Hasting weights are defined as:

$$a_{ij} = \begin{cases} \frac{1}{1+\max\{d_i, d_j\}} & \text{if } (i,j) \in E \text{ and } i \neq j \\ 1 - \sum_{h \in \mathcal{N}_i \setminus \{i\}} a_{ih} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The matrix $\boldsymbol{A}$ of Metropolis-Hasting weights is symmetric and doubly stochastic.

1.2.3 Time-varying digraphs

Time-varying digraph Graph $G = (I, E(k))$ that changes at each iteration k. It can be described by a sequence $\{G(k)\}_{k\geq 0}$.

Time-varying digraph

Jointly strongly connected digraph Time-varying digraph that is asymptotically strongly connected:

Jointly strongly connected digraph

$$\forall k \geq 0 : \bigcup_{\tau=k}^{+\infty} G(\tau) \text{ is strongly connected}$$

Uniformly jointly strongly/B-strongly connected digraph Time-varying digraph that is strongly connected in B steps:

Uniformly jointly strongly/B-strongly connected digraph

$$\forall k \geq 0, \exists B \in \mathbb{N} : \bigcup_{\tau=k}^{k+B} G(\tau) \text{ is strongly connected}$$

Remark. (Uniformly) jointly strongly connected digraph can be disconnected at some time steps k.

Averaging distributed algorithm Given a time-varying digraph $\{G(k)\}_{k\geq 0}$ (always with self-loops), in- and out-neighbors distributed algorithms can be formulated as:

Averaging distributed algorithm over time-varying digraph

$$x_i^{k+1} = \sum_{j \in \mathcal{N}_i^{\text{IN}}(k)} a_{ij}(k) x_j^k \quad x_i^{k+1} = \sum_{j \in \mathcal{N}_i^{\text{OUT}}(k)} a_{ij}(k) x_j^k$$

Linear time-varying (LTV) discrete-time system In matrix form, it can be formulated as:

$$x^{k+1} = \boldsymbol{A}(k)x^k$$

Linear time-varying (LTV) discrete-time system

Theorem 1.2.3 (Discrete-time consensus over time-varying graphs)**.** Consider a time-varying discrete-time average system with digraphs $\{G(k)\}_{k\geq 0}$ (all with self-loops) and weighted adjacency matrices $\{\boldsymbol{A}(k)\}_{k\geq 0}$. Assume:

Discrete-time consensus over time-varying graphs

- Each non-zero edge weight $a_{ij}(k)$, self-loops included, are larger than a constant $\varepsilon > 0$,
- There exists $B \in \mathbb{N}$ such that $\{G(k)\}_{k\geq 0}$ is B-strongly connected.

It holds that there exists a vector $\mathbf{w} \in \mathbb{R}^N$, $\mathbf{w} > 0$ such that the consensus converges to:

$$\lim_{k\to\infty} x^k = \mathbf{1}\frac{\mathbf{w}^T x^0}{\mathbf{w}^T \mathbf{1}}$$

Moreover, if each $\boldsymbol{A}(k)$ is doubly stochastic, it holds that the consensus is the average:

$$\lim_{k\to\infty} x^k = \mathbf{1}\frac{1}{N}\sum_{i=1}^{N} x_i^0$$